Non-linear structure in the Universe
Cosmology on the Beach

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Limited options

- Beyond a certain scale, linear perturbation theory breaks down
  - Definition of “non-linear scale”?
- At this point we have few options:
  - Analytical models of non-linear growth.
    - Zel’dovich approximation.
    - Spherical top-hat collapse.
  - Perturbation theory.
    - Realm of validity? Convergence criterion?
    - Good for small corrections to almost linear problems.
  - Direct simulation.
    - Numerical convergence.
    - What models to run?
    - Missing physics.

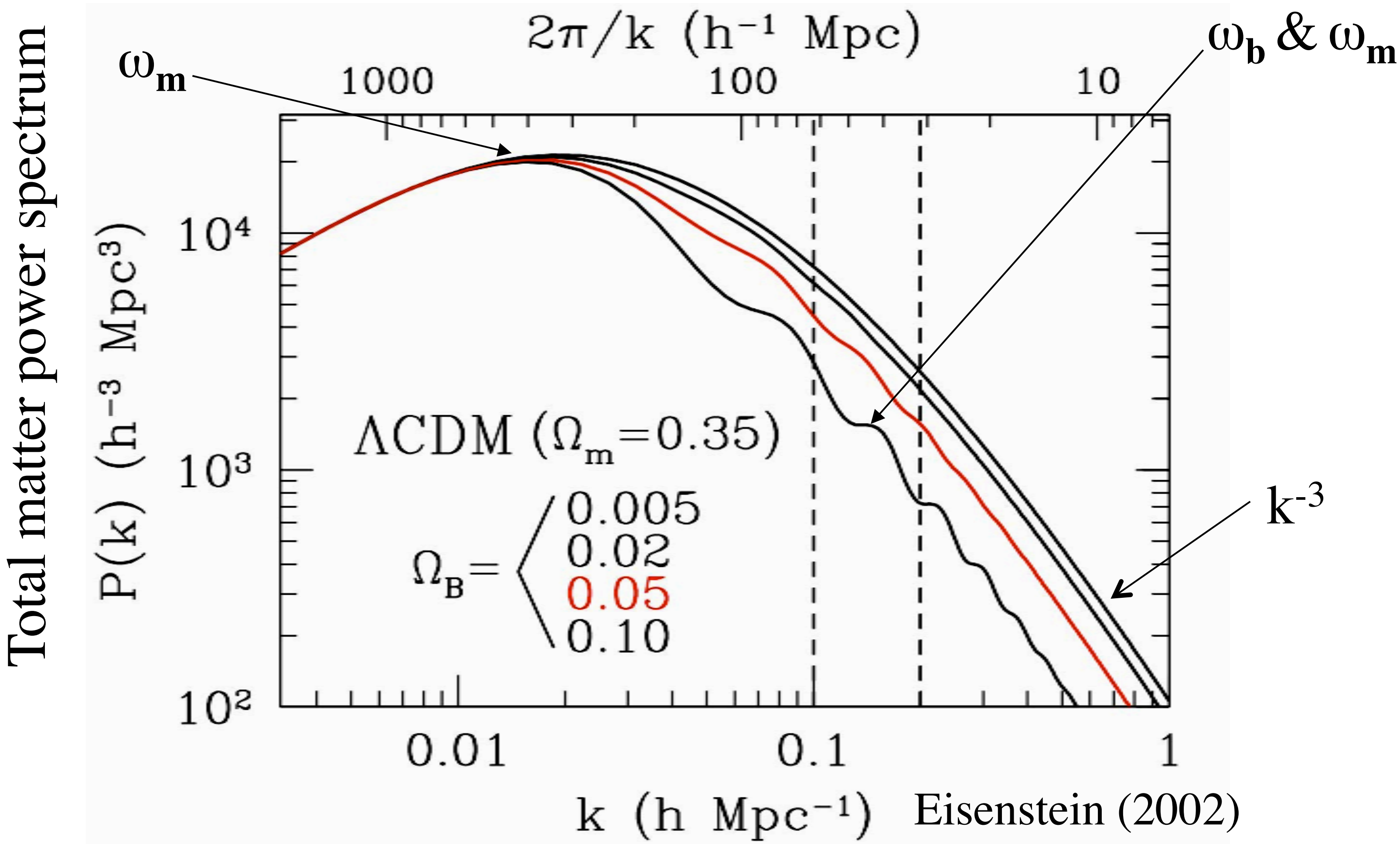
Notation

$$
\begin{aligned}
\delta(\mathbf{x}) &= \frac{\rho(\mathbf{x}) - \bar{\rho}}{\bar{\rho}} = \frac{\delta\rho}{\rho}(\mathbf{x}) \\
\delta(\mathbf{k}) &= \int d^3x \; \delta(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}} \\
\langle \delta(\mathbf{k})\delta^\star(\mathbf{k}') \rangle &= (2\pi)^3 \delta_D(\mathbf{k} - \mathbf{k}') P(k) \\
\Delta^2(k) &= \frac{k^3 P(k)}{2\pi^2} \\
\xi(x) &= \int \frac{d^3k}{(2\pi)^3} P(k) e^{i\mathbf{k}\cdot\mathbf{x}} \\
&= \int \frac{dk}{k} \; \Delta^2(k) j_0(kr)
\end{aligned}
$$

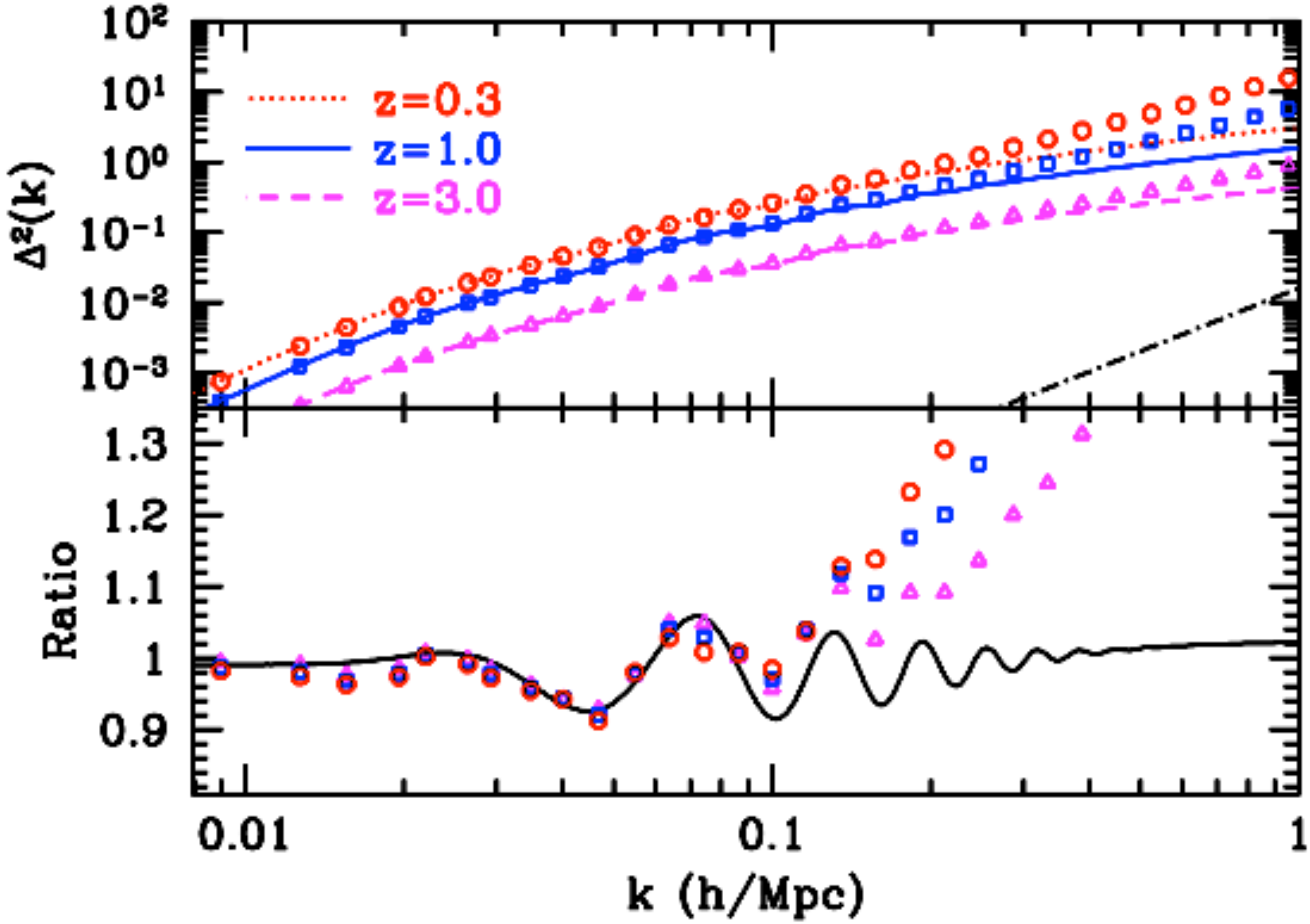
Linear PT

- For many scales and most of age of Universe linear perturbation theory is valid.
- Transfer function, *T(k)*, encodes 14Gyr of evolution.
  - $\delta_{today}(k) \sim (\text{growth}) \times T(k)\delta_{init}(k)$.
  - Main features RD->MD->ΛD.
  - Structure only grows when matter dominates energy density of Universe.

Matter power spectrum: $P_L(k)$



Non-linearity



Scale of non-linearity

- There are several ways to define a "scale" of non-linearity.
- Where $\Delta^2(k)$=1 (or ½, or …).
  - Dangerous when $\Delta^2(k)$ is very flat.
- By the rms linear theory displacement.

$$R_{\rm nl}^2 \propto \frac{1}{k_{\rm nl}^2} \propto \int \frac{dk}{k}\ \frac{\Delta^2(k)}{k^2} \propto \int dk\ P(k)$$

- Where the 2nd order correction to some quantity is 1% (10%) of the 1st order term.

“Weak” non-linearity: Cosmological perturbation theory

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Perturbation theory

- There is no reason (in principle) to stop at linear order in perturbation theory.
  - Can expand to all orders: $\delta=\delta^{(1)}+\delta^{(2)}+\delta^{(3)}+\ldots$
  - Can sum subsets of terms.
  - Usefulness/convergence of such an expansion not always clear.
- Consider **only** dark matter and **assume** we are in the single-stream limit.

Peebles (1980), Juszkiewicz (1981), Goroff++(1986), Makino++(1992), Jain&Bertschinger(1994), Fry (1994).
Reviews/comparison with N-body:
Bernardeau++(2002; Phys. Rep. 367, 1).
Carlson++(2009; PRD 80, 043531)

Equations of motion

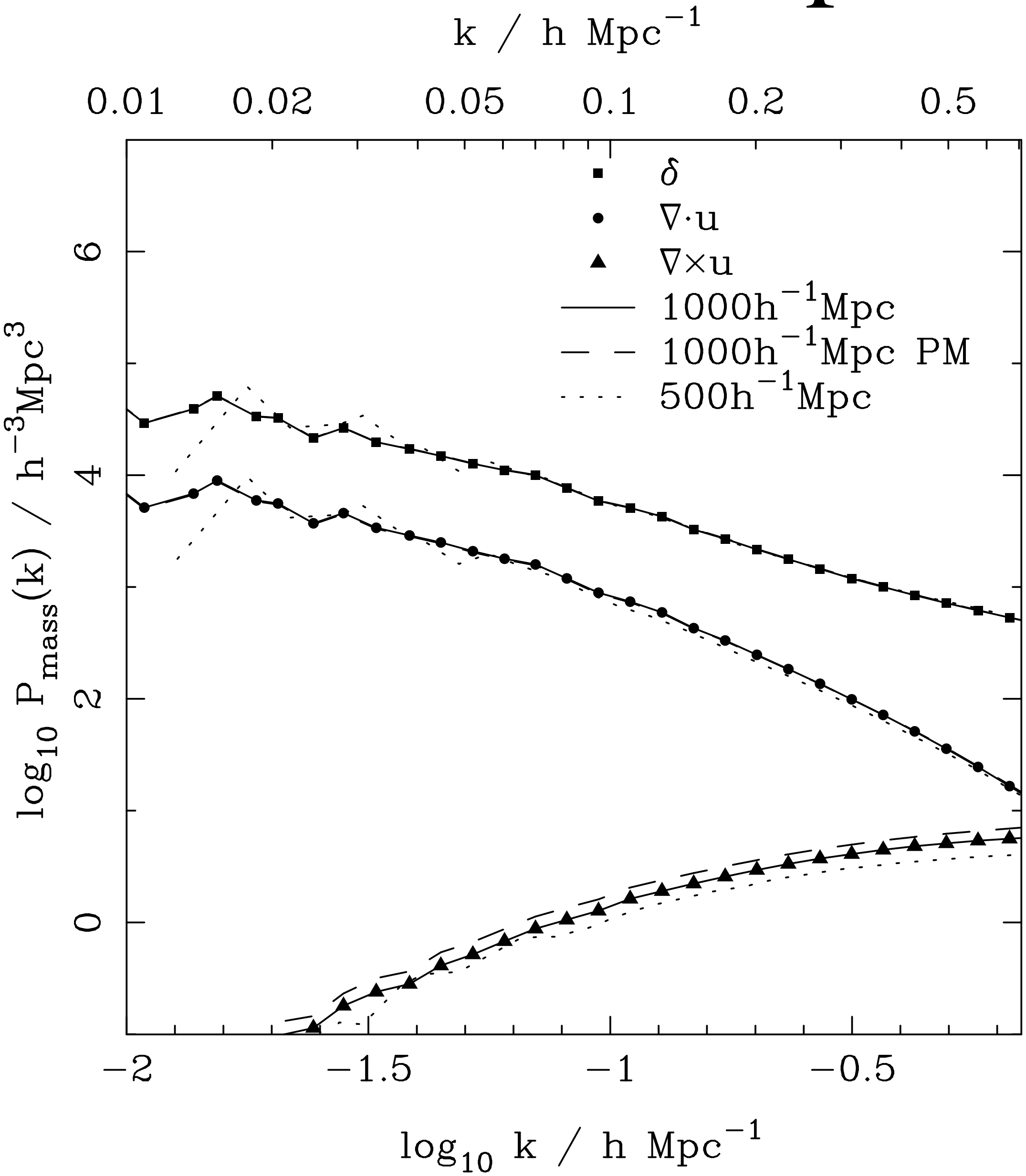
Under these approximations, and assuming Ω_m=1

$$
\begin{aligned}
\frac{\partial \delta}{\partial \tau} + \vec{\nabla} \cdot \left[(1+\delta)\vec{v}\right] &= 0 \\
\frac{\partial \vec{v}}{\partial \tau} + \mathcal{H}\vec{v} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{v} &= -\vec{\nabla}\Phi \\
\nabla^2 \Phi &= \tfrac{3}{2}\mathcal{H}^2 \delta
\end{aligned}
$$

Gauge?

- Very familiar looking fluid equations
  - means we can borrow methods/ideas from other fields.
- Note the quadratic nature of the non-linearity.
- Since equations are now non-linear, can't use super-position of (exact) solutions even if they could be found!
- Proceed by perturbative expansion.

Velocities are ≈ potential flow



Percival & White (2009)

Assume that *v* comes from a potential flow (self-consistent; curl[v] ~a^{-1} at linear order) then it is totally specified by its divergence, θ.

Go into Fourier space

Putting the quadratic terms on the rhs and going into Fourier space:

Div

$\mathbf{v}$~$(\mathbf{q}/\mathrm{q}^2)\theta$

Product= Convolution

$$
\begin{aligned}
\frac{\partial\delta(\vec{k})}{\partial\tau} + \theta(\vec{k}) &= -\int \frac{d^3q}{(2\pi)^3}\,\frac{\vec{k}\cdot\vec{q}}{q^2}\,\overline{\theta(\vec{q})\delta(\vec{k}-\vec{q})}, \\
\frac{\partial\theta(\vec{k})}{\partial\tau} + \mathcal{H}\theta(\vec{k}) + \frac{3}{2}\Omega_m\mathcal{H}^2\delta(\vec{k}) &= -\int \frac{d^3q}{(2\pi)^3}\,\frac{k^2\ \vec{q}\cdot(\vec{k}-\vec{q})}{2q^2|\vec{k}-\vec{q}|^2} \\
&\times\quad \theta(\vec{q})\theta(\vec{k}-\vec{q}).
\end{aligned}
$$

Linear order

- To lowest order in δ and θ:

$$
\begin{aligned}
\delta_L(\mathbf{k}, z) &= \frac{D(z)}{D(z_i)}\, \delta_i(\mathbf{k}) \\
\theta_L(\mathbf{k}, z) &= -f(z)\mathcal{H}(z)\, \frac{D(z)}{D(z_i)}\, \delta_i(\mathbf{k})
\end{aligned}
$$

- with $f(z) \sim \Omega_m^{0.6} = 1$ for $\Omega_m = 1$ and $D(a) \sim a$.
- Decaying mode, $\delta \sim a^{-3/2}$, has to be zero for δ to be well-behaved as a->0.
- Define $\delta_0 = \delta_L(k, z=0)$.

Standard perturbation theory

- Develop δ and θ as power series:

$$
\begin{aligned}
\delta(\mathbf{k}) &= \sum_{n=1}^{\infty} a^n \delta^{(n)}(\mathbf{k}) \\
\theta(\mathbf{k}) &= -\mathcal{H} \sum_{n=1}^{\infty} a^n \theta^{(n)}(\mathbf{k})
\end{aligned}
$$

- then the $\delta^{(n)}$ can be written

$$
\begin{aligned}
\delta^{(n)}(\mathbf{k}) &= \int \frac{d^3q_1 d^3q_2 \cdots d^3q_n}{(2\pi)^{3n}} (2\pi)^3 \delta_D\left(\sum \mathbf{q}_i - \mathbf{k}\right) \\
&\times \quad F_n\left(\{\mathbf{q}_i\}\right) \delta_0(\mathbf{q}_1) \cdots \delta_0(\mathbf{q}_n)
\end{aligned}
$$

- with a similar expression for $\theta^{(n)}$.
- The F_n and G_n are just ratios of dot products of the ***q***s and obey simple recurrence relations.

Recurrence relations I

- Plugging the expansion into our equations and using
  - $(d/d\tau)a^n = nHa^n$
  - $(d/d\tau)H = (-1/2)H^2$ for EdS
- we have (canceling H from both sides):

$$n\delta^{(n)} + \theta^{(n)} = -\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}(2\pi)^3\delta(\vec{k}-\vec{q}_1-\vec{q}_2)\frac{\vec{k}\cdot\vec{q}_1}{q_1^2}\sum_{m=1}^{n-1}\theta_m(\vec{q}_1)\delta_{n-m}(\vec{q}_2)$$

$$3\delta^{(n)} + (2n+1)\theta^{(n)} = -\int \frac{d^3q_1}{(2\pi)^3}\frac{d^3q_2}{(2\pi)^3}(2\pi)^3\delta(\vec{k}-\vec{q}_1-\vec{q}_2)\frac{k^2(\vec{q}_1\cdot\vec{q}_2)}{q_1^2q_2^2}\sum_{m=1}^{n-1}\theta_m(\vec{q}_1)\theta_{n-m}(\vec{q}_2)$$

Recurrence relations II

- Which we can rewrite

$$\delta^{(n)} = \frac{(2n+1)A_n - B_n}{(2n+3)(n-1)} \quad , \quad \theta^{(n)} = \frac{-3A_n + nB_n}{(2n+3)(n-1)}$$

- where A_n and B_n are the rhs mode-coupling integrals.
- This generates recursion relations for the F_n and G_n (because of the sums in A_n and B_n)

$$F_n = \sum_{m=1}^{n-1} \frac{G_m}{(2n+3)(n-1)} \left[(2n+1)\frac{\vec{k}\cdot\vec{k}_1}{k_1^2} F_{n-m} + \frac{k^2(\vec{k}_1\cdot\vec{k}_2)}{k_1^2 k_2^2} G_{n-m} \right]$$

$$G_n = \sum_{m=1}^{n-1} \frac{G_m}{(2n+3)(n-1)} \left[3\frac{\vec{k}\cdot\vec{k}_1}{k_1^2} F_{n-m} + n\frac{k^2(\vec{k}_1\cdot\vec{k}_2)}{k_1^2 k_2^2} G_{n-m} \right]$$

Example: 2nd order

- The coupling function:

$$F_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{5}{7} + \frac{2}{7}\frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} + \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)}{2}\left(k_1^{-2} + k_2^{-2}\right)$$

- where we have symmetrized the function in terms of its arguments.
  - Note: this function peaks when $k_1 \sim k_2 \sim k/2$.
  - This will be important later.

Formal development

- We can make the expressions above more formal by defining $\eta=\ln(a)$ and

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = e^{-\eta} \begin{pmatrix} \delta \\ -\theta/\mathcal{H} \end{pmatrix}$$

- then writing

$$\partial_\eta \phi_a = -\Omega_{ab}\phi_b + e^\eta \gamma_{abc}\phi_b\phi_c$$

- with the obvious definitions of Ω and γ.
- We can also define $P \sim \langle\phi\phi\rangle$, $B \sim \langle\phi\phi\phi\rangle$ so e.g.

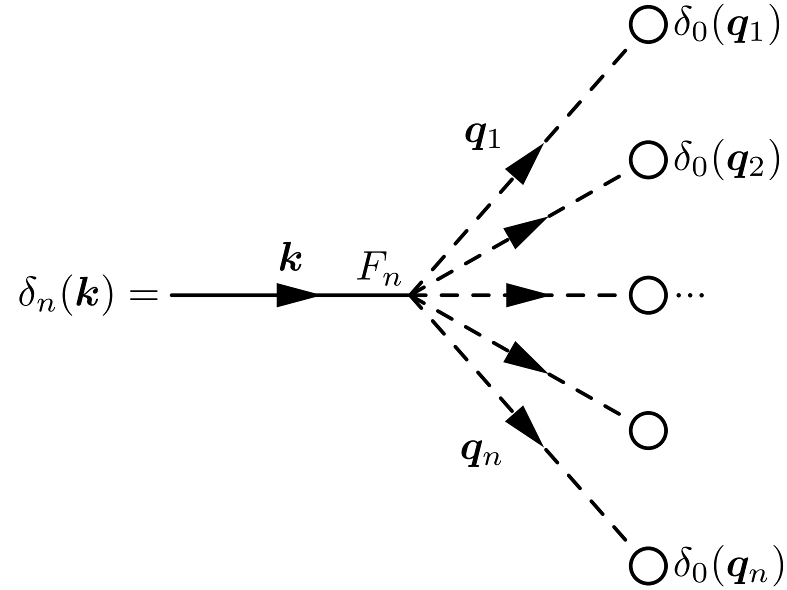
$$\partial_\eta P_{ab} = -\Omega_{ac}P_{cb} - \Omega_{bc}P_{ac} + e^\eta \int d^3q \left[\gamma_{acd}B_{bcd} + B_{acd}\gamma_{bcd}\right]$$

Power spectrum

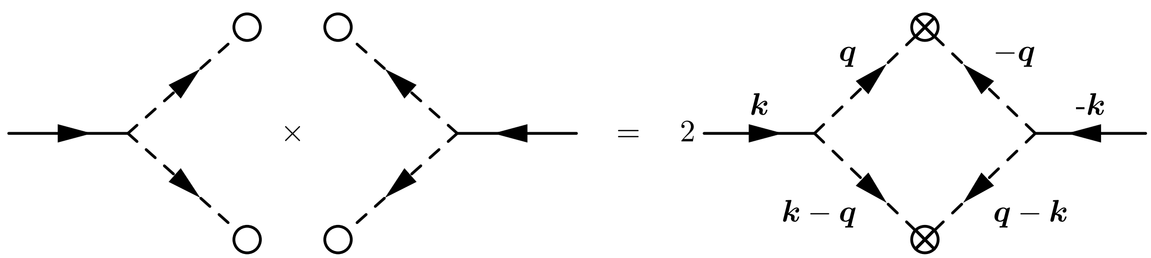
- If the initial fluctuations are Gaussian only expectation values even in δ_0 survive:
  - $P(k) \sim \langle[\delta^{(1)}+\delta^{(2)}+\delta^{(3)}+\ldots][\delta^{(1)}+\delta^{(2)}+\delta^{(3)}+\ldots]\rangle$
  - $= P^{(1,1)} + 2P^{(1,3)} + P^{(2,2)} + \cdots$
- with terms like $\langle\delta^{(1)}\delta^{(2)}\rangle$ vanishing because they reduce to $\langle\delta_0\delta_0\delta_0\rangle$.

Perturbation theory: diagrams

Just as there is a diagrammatic short-hand for perturbation theory in quantum field theory, so there is in cosmology:



$\equiv \quad (2\pi)^3\delta_D(\boldsymbol{q}+\boldsymbol{q}')P_0(q),$



$$= 2\int \frac{d^3q}{(2\pi)^3} F_2(\boldsymbol{q},\boldsymbol{k}-\boldsymbol{q})F_2(-\boldsymbol{q},\boldsymbol{q}-\boldsymbol{k})P_0(q)P_0(|\boldsymbol{k}-\boldsymbol{q}|)$$

Example: 2nd order

$$P^{(1,3)}(k) = \frac{1}{252}\frac{k^3}{4\pi^2}P_L(k)\int_0^\infty dr\ P_L(kr)\left[\frac{12}{r^2} - 158 + 100r^2 - 42r^4 + \frac{3}{r^2}(r^2-1)^3(7r^2+2)\ln\left|\frac{1+r}{1-r}\right|\right],$$

$$P^{(2,2)}(k) = \frac{1}{98}\frac{k^3}{4\pi^2}\int_0^\infty dr\ P_L(kr)\int_{-1}^{1} dx\ P_L\left(k\sqrt{1+r^2-2rx}\right) \times \frac{(3r+7x-10rx^2)^2}{(1+r^2-2rx)^2}.$$

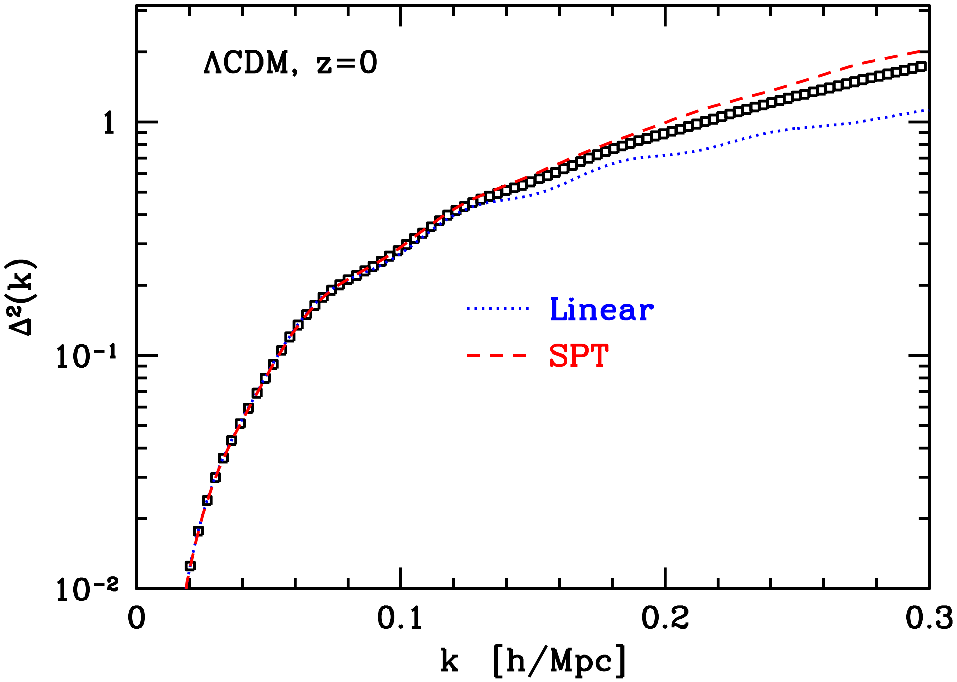
Perturbation theory enables the generation of truly impressive looking equations which arise from simple angle integrals. Like Feynman integrals, they are simple but look erudite!

Example: 2^{nd} order

- At low *k*, $P^{(2,2)}$ is positive and $P^{(1,3)}$ is negative
  - Large cancellation.
- For large *k* total contribution is negative:
  - $P^{(2,2)} \sim (1/4)\, k^2 \Sigma^2\, P_L(k)$
  - $P^{(1,3)} \sim -(1/2)\, k^2 \Sigma^2\, P_L(k)$
- Here Σ is the rms displacement (in each component) in linear theory.
  - It will come up again!!

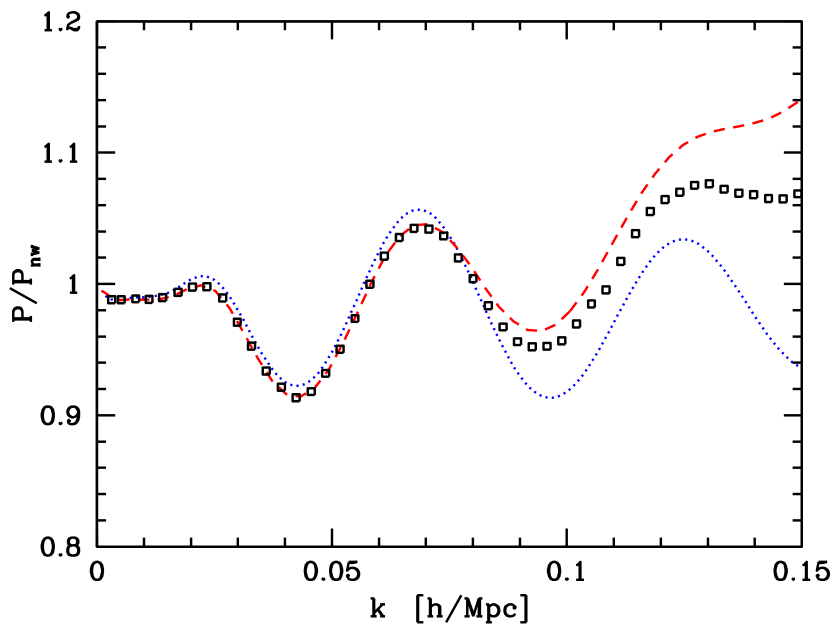
$$\Sigma^2 = \frac{1}{3\pi^2} \int_0^\infty dq\ P_L(q)$$

Example



The lowest order correction to the matter power spectrum at z=0 (1-loop SPT).

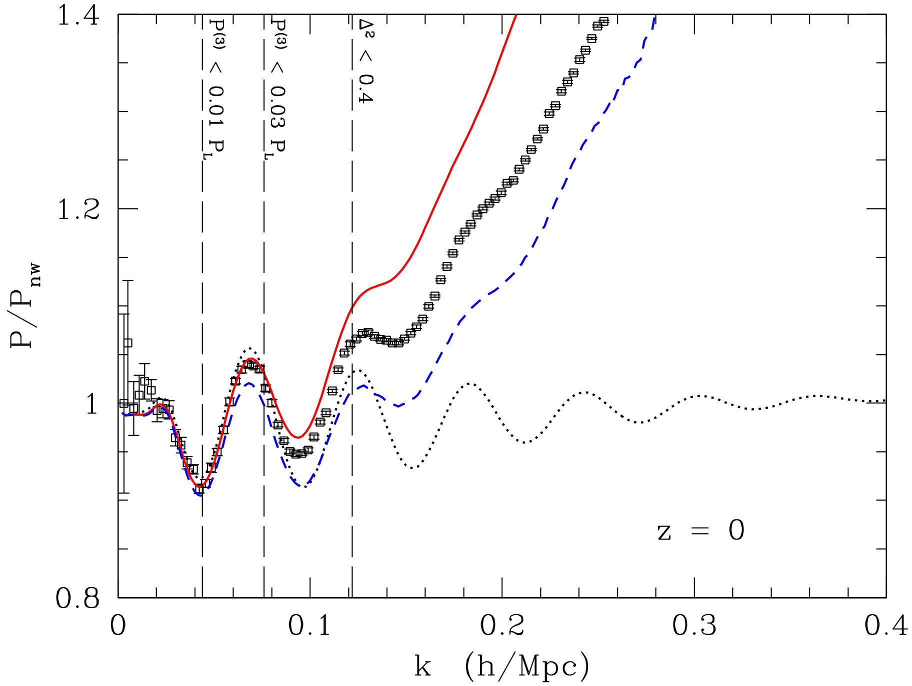
Note the improvement at low k where non-linear growth causes a suppression of power (pre-virialization).



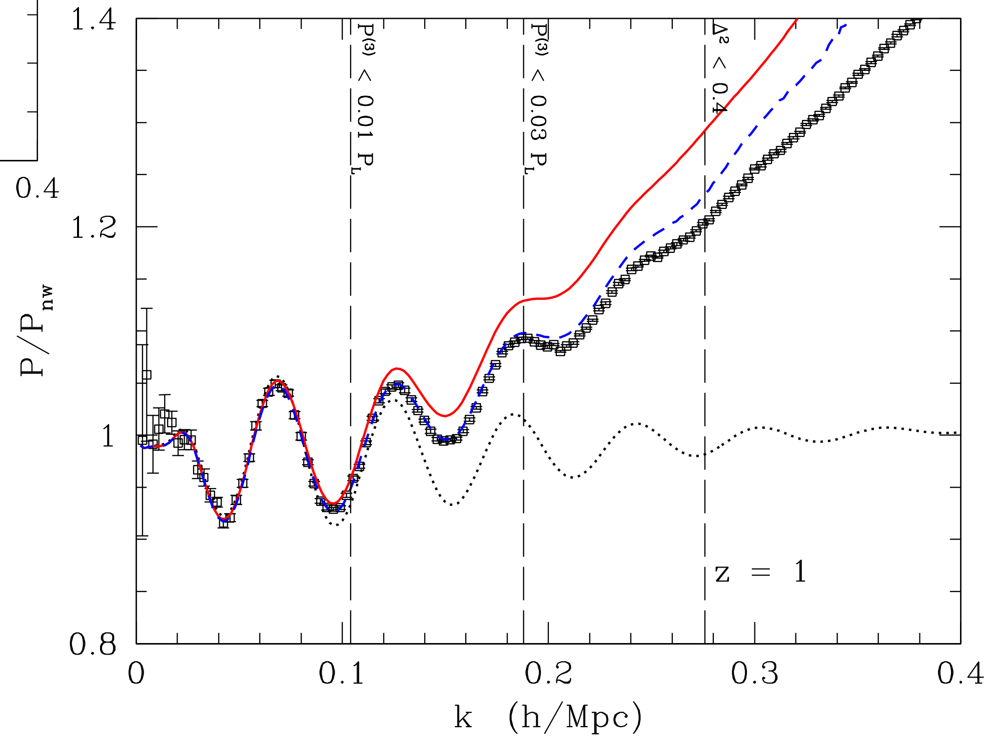
Beyond 2^{nd} order

- Expressions for higher orders are easy to derive, especially using computer algebra packages.
- Using rotation symmetry the N^{th} order contribution requires mode coupling integrals of dimension 3N-1.
  - Best done using Monte-Carlo integration.
  - Prohibitive for very high orders.
  - Not clear this expansion is converging!

Comparison with exact results



Broad-band shape of P_L has been divided out to focus on more subtle features.



Linear

1st order correction

2nd order correction

Carlson++09

Including bias

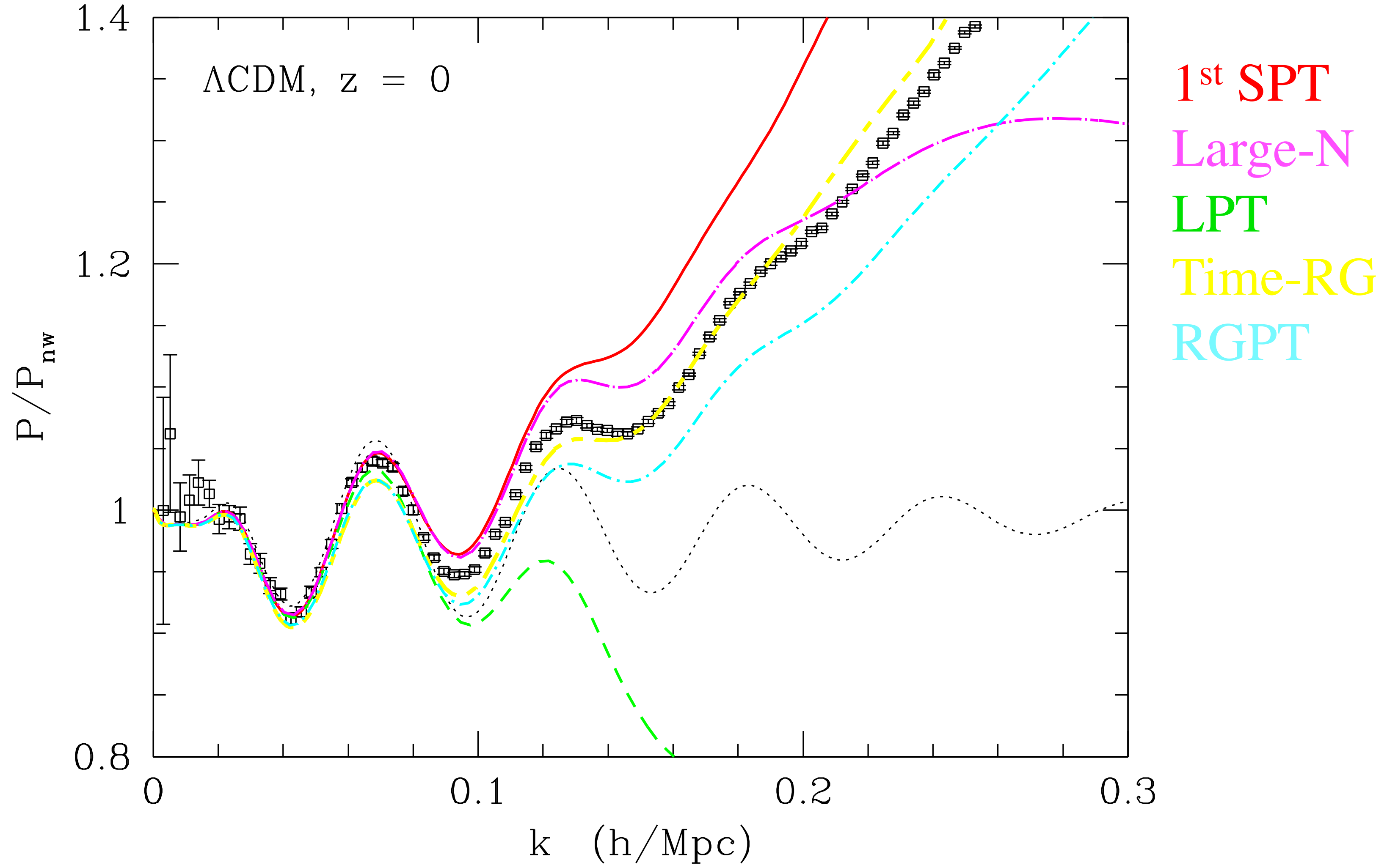
- Perturbation theory clearly cannot describe the formation of collapsed, bound objects such as dark matter halos.
- We can extend the usual thinking about “linear bias” to a power-series in the Eulerian density field:
  - $\delta_{obj} = \Sigma\, b_n(\delta^n/n!)$
- The expressions for P(k) now involve b_1 to lowest order, b_1 and b_2 to next order, etc.
  - The physical meaning of these terms is actually hard to figure out, and the validity of the defining expression is dubious, but this is the standard way to include bias in Eulerian perturbation theory.

Other methods

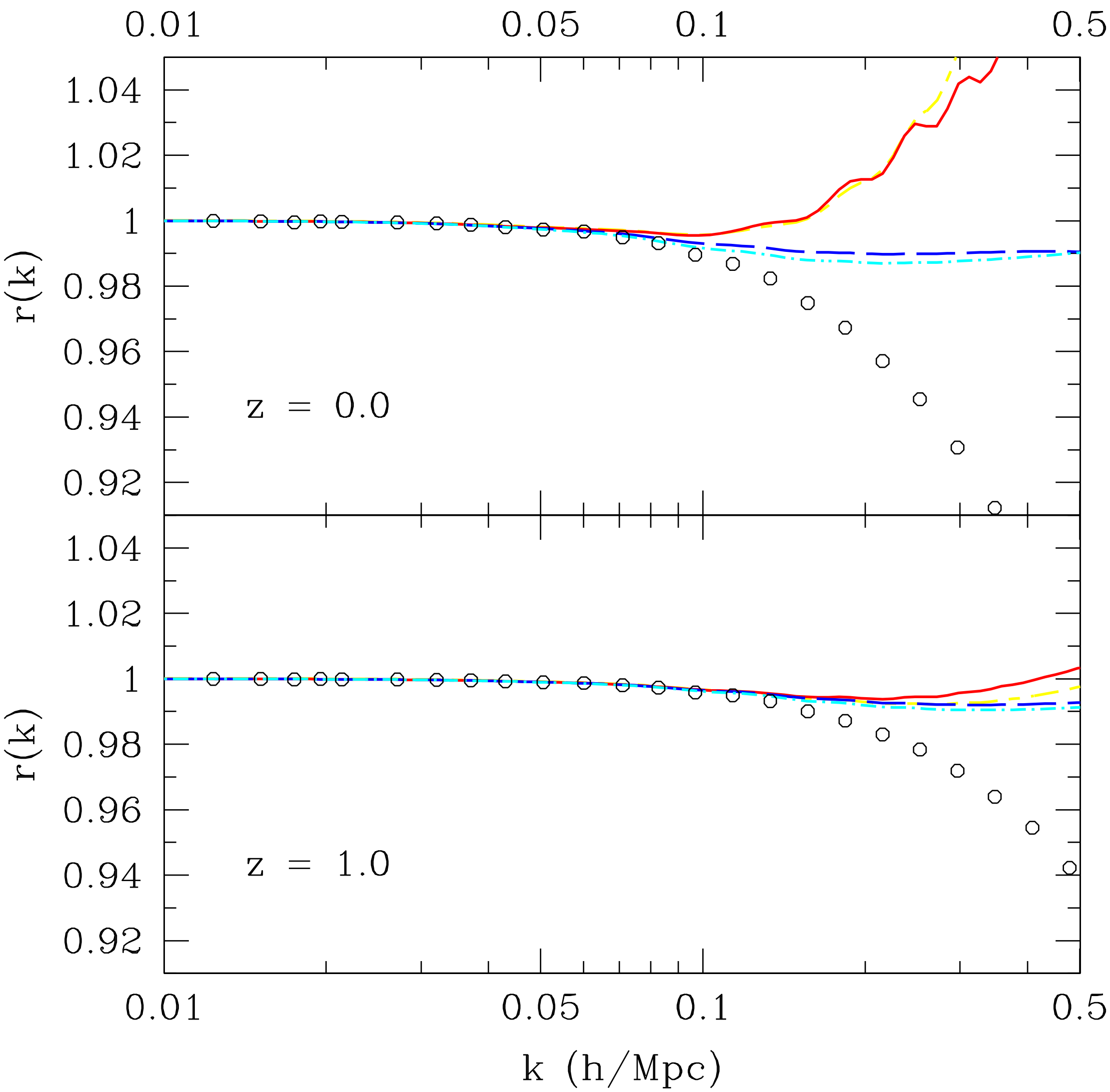
- Renormalized perturbation theory
  - A variant of “Dyson-Wyld” resummation.
  - An expansion in “order of complexity”.
- Closure theory
  - Write expressions for $(d/d\tau)P$ in terms of P, B, T, …
  - Approximate B by leading-order expression in SPT.
- Time-RG theory (& RGPT)
  - As above, but assume T=0
  - Good for models with $m_\nu>0$ where linear growth is scale-dependent.
- Path integral formalism
  - Perturbative evaluation of path integral gives SPT.
  - Large N expansion, 2PI effective action, steepest descent.
- Lagrangian perturbation theory

(see Carlson++09 for references)

Some other theories



Other statistics



PT makes predictions for other statistics as well. For example, the power spectra of the velocity and the density-velocity cross spectrum. Here it seems to do less well.

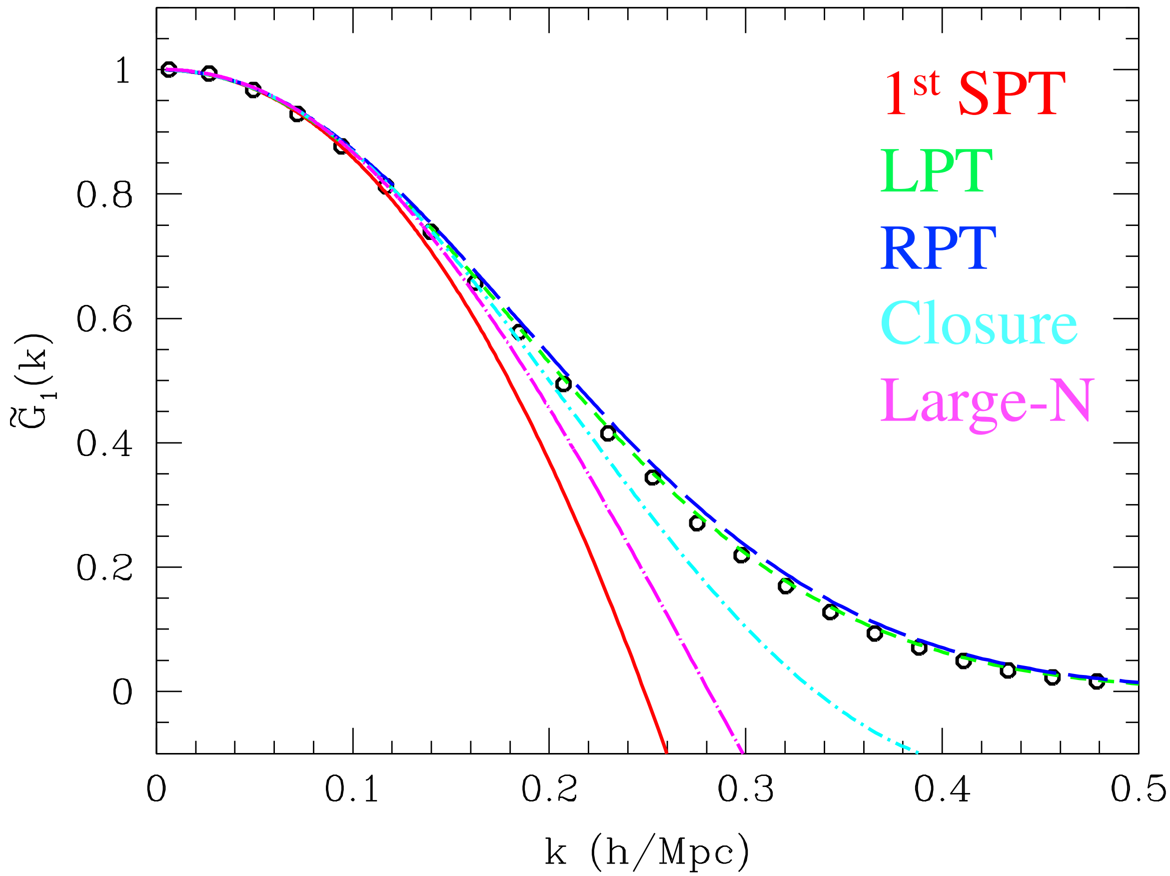
SPT

RPT

Closure

Time-RG

Some other quantities



The propagator, or

$$G(k) \propto \frac{\langle \delta_{NL} \delta_L^* \rangle}{\langle \delta_L \delta_L^* \rangle}$$

which measures the decoherence of the final density field due to non-linear evolution.

Carlson++09

Lagrangian perturbation theory

- A different approach to PT, which has been radically developed recently by Matsubara and is *very* useful for BAO.
  - Buchert89, Moutarde++91, Bouchet++92, Catelan95, Hivon++95.
  - Matsubara (2008a; PRD, 77, 063530)
  - Matsubara (2008b; PRD, 78, 083519)
- Relates the current (Eulerian) position of a mass element, **x**, to its initial (Lagrangian) position, **q**, through a displacement vector field, Ψ.

Lagrangian perturbation theory

$$
\begin{aligned}
\delta(\mathbf{x}) &= \int d^3q \; \delta_D(\mathbf{x} - \mathbf{q} - \mathbf{\Psi}) - 1 \\
\delta(\mathbf{k}) &= \int d^3q \; e^{-i\mathbf{k}\cdot\mathbf{q}} \left(e^{-i\mathbf{k}\cdot\mathbf{\Psi}(\mathbf{q})} - 1 \right) \; .
\end{aligned}
$$

$$
\frac{d^2\mathbf{\Psi}}{dt^2} + 2H\frac{d\mathbf{\Psi}}{dt} = -\nabla_x \phi \left[\mathbf{q} + \mathbf{\Psi}(\mathbf{q})\right]
$$

$$
\begin{aligned}
\mathbf{\Psi}^{(n)}(\mathbf{k}) &= \frac{i}{n!} \int \prod_{i=1}^{n} \left[\frac{d^3k_i}{(2\pi)^3} \right] \; (2\pi)^3 \delta_D \left(\sum_i \mathbf{k}_i - \mathbf{k} \right) \\
&\times \; \mathbf{L}^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n, \mathbf{k}) \delta_0(\mathbf{k}_1) \cdots \delta_0(\mathbf{k}_n)
\end{aligned}
$$

Kernels

$$\mathbf{L}^{(1)}(\mathbf{p}_1) = \frac{\mathbf{k}}{k^2} \tag{1}$$

$$\mathbf{L}^{(2)}(\mathbf{p}_1, \mathbf{p}_2) = \frac{3}{7}\frac{\mathbf{k}}{k^2}\left[1 - \left(\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{p_1 p_2}\right)^2\right] \tag{2}$$

$$\mathbf{L}^{(3)}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3) = \cdots \tag{3}$$

$$\mathbf{k} \equiv \mathbf{p}_1 + \cdots + \mathbf{p}_n$$

Standard LPT

- If we expand the exponential and keep terms consistently in δ_0 we regain a series $\delta=\delta^{(1)}+\delta^{(2)}+\ldots$ where $\delta^{(1)}$ is linear theory and e.g.

$$\begin{aligned}\delta^{(2)}(\mathbf{k}) &= \frac{1}{2}\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta_D(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k})\delta_0(\mathbf{k}_1)\delta_0(\mathbf{k}_2) \\ &\times \left[\mathbf{k}\cdot\mathbf{L}^{(2)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k})+\mathbf{k}\cdot\mathbf{L}^{(1)}(\mathbf{k}_1)\mathbf{k}\cdot\mathbf{L}^{(1)}(\mathbf{k}_2)\right]\end{aligned}$$

- which regains "SPT".
  - The quantity in square brackets is F_2.

$$F_2(\mathbf{k}_1,\mathbf{k}_2)=\frac{5}{7}+\frac{2}{7}\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)^2}{k_1^2k_2^2}+\frac{(\mathbf{k}_1\cdot\mathbf{k}_2)}{2}\left(k_1^{-2}+k_2^{-2}\right)$$

LPT power spectrum

- Alternatively we can use the expression for $\delta_{\mathbf{k}}$ to write

$$P(k) = \int d^3q \; e^{-i\vec{k}\cdot\vec{q}} \left(\left\langle e^{-i\vec{k}\cdot\Delta\vec{\Psi}} \right\rangle - 1 \right)$$

- where $\Delta\Psi=\Psi(\mathbf{q})-\Psi(0)$.
- Expanding the exponential and plugging in for $\Psi^{(n)}$ gives the usual results.
- **BUT** Matsubara suggested a different and very clever approach.

Cumulants

- The cumulant expansion theorem allows us to write the expectation value of the exponential in terms of the exponential of expectation values.
- Expand the terms $(\mathbf{k}\Delta\Psi)^N$ using the binomial theorem.
- There are two types of terms:
  - Those depending on Ψ at same point.
    - This is independent of position and can be factored out of the integral.
  - Those depending on Ψ at different points.
    - These can be expanded as in the usual treatment.

Example

- Imagine Ψ is Gaussian with mean zero.
- For such a Gaussian: $\langle e^{\Psi}\rangle = \exp[\sigma^2/2]$.

$$P(k) = \int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} \left(\left\langle e^{-ik_i \Delta\Psi_i(\mathbf{q})} \right\rangle - 1 \right)$$

$$\left\langle e^{-i\mathbf{k}\cdot\Delta\boldsymbol{\Psi}(q)} \right\rangle = \exp\left[-\frac{1}{2} k_i k_j \left\langle \Delta\Psi_i(\mathbf{q}) \Delta\Psi_j(\mathbf{q}) \right\rangle \right]$$

$$k_i k_j \left\langle \Delta\Psi_i(\mathbf{q}) \Delta\Psi_j(\mathbf{q}) \right\rangle = 2k_i^2 \langle \Psi_i^2(\mathbf{0}) \rangle - 2k_i k_j \xi_{ij}(\mathbf{q})$$

Keep exponentiated.

Expand

Resummed LPT

- The first corrections to the power spectrum are then:

$$P(k) = e^{-(k\Sigma)^2/2}\left[P_L(k) + P^{(2,2)}(k) + \widetilde{P}^{(1,3)}(k)\right],$$

- where $P^{(2,2)}$ is as in SPT but part of $P^{(1,3)}$ has been "resummed" into the exponential prefactor.

- The exponential prefactor is identical to that obtained from
  - The peak-background split (Eisenstein++07)
  - Renormalized Perturbation Theory (Crocce++08).

Beyond real-space mass

- One of the more impressive features of Matsubara's approach is that it can gracefully handle both biased tracers and redshift space distortions.
- In redshift space, in the plane-parallel limit,

$$\mathbf{\Psi} \to \mathbf{\Psi} + \frac{\widehat{\mathbf{z}} \cdot \dot{\mathbf{\Psi}}}{H} \,\widehat{z} = R\,\mathbf{\Psi}$$

- In PT $\mathbf{\Psi}^{(n)} \propto D^n \Rightarrow R_{ij}^{(n)} = \delta_{ij} + nf\,\widehat{z}_i\widehat{z}_j$

- Again we're going to leave the zero-lag piece exponentiated so that the prefactor contains

$$k_i k_j R_{ia} R_{jb} \delta_{ab} = (k_a + fk\mu\widehat{z}_a)\,(k_a + fk\mu\widehat{z}_a) = k^2\left[1 + f(f+2)\mu^2\right]$$

- while the $\xi(r)$ piece, when FTed, becomes the usual Kaiser expression plus higher order terms.

Beyond real-space mass

- One of the more impressive features of Matsubara's approach is that it can gracefully handle both biased tracers and redshift space distortions.

- For bias local in Lagrangian space:

$$\delta_{\rm obj}(\mathbf{x}) = \int d^3q \; F\left[\delta_L(\mathbf{q})\right] \delta_D(\mathbf{x} - \mathbf{q} - \mathbf{\Psi})$$

- we obtain

$$P(k) \quad = \quad \int d^3q \; e^{-i\mathbf{k}\cdot\mathbf{q}} \left[\int \frac{d\lambda_1}{2\pi} \frac{d\lambda_2}{2\pi} F(\lambda_1) F(\lambda_2) \left\langle e^{i[\lambda_1 \delta_L(\mathbf{q}_1) + \lambda_2 \delta_L(\mathbf{q}_2)] + i\mathbf{k}\cdot\Delta\mathbf{\Psi}} \right\rangle - 1 \right]$$

- which can be massaged with the same tricks as we used for the mass.
- If we assume halos/galaxies form at peaks of the initial density field ("peaks bias") then explicit expressions for the integrals of F exist.

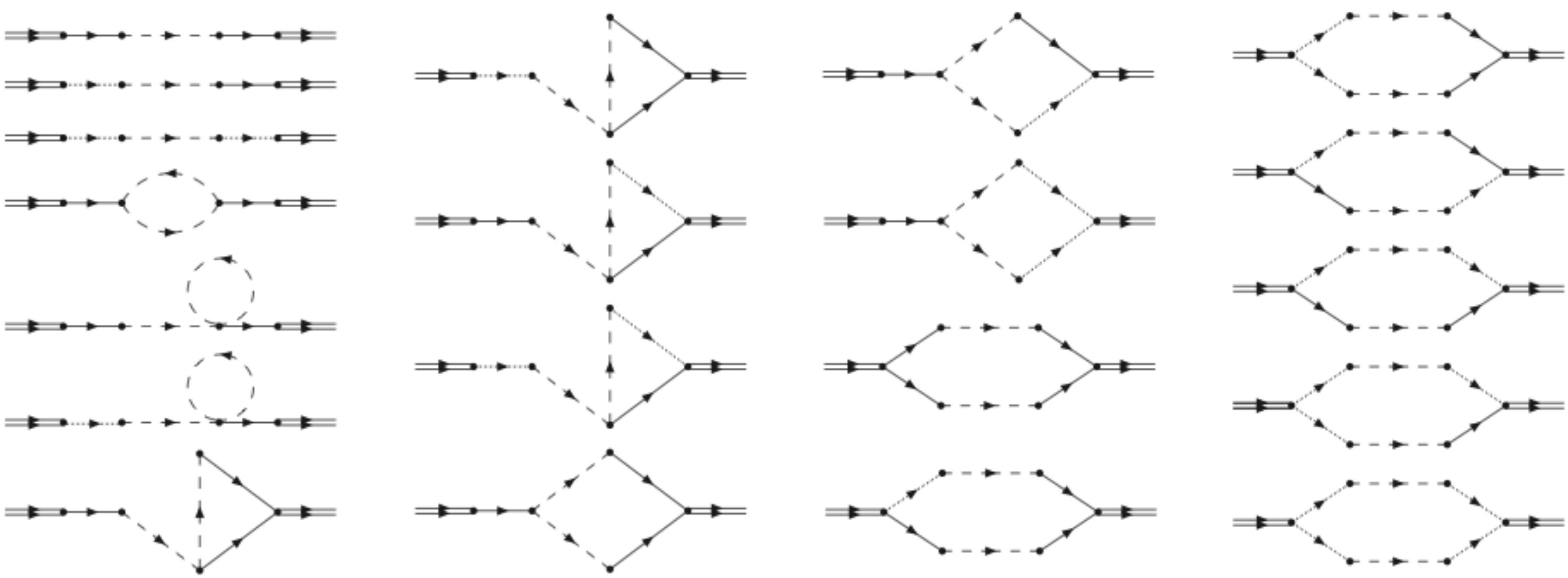
The answer

$$P_{\rm obj}^{(s)} = e^{-\left[1+f(f+2)\mu^2\right]k^2\Sigma^2/2}\left[\left(b+f\mu^2\right)^2 P_L + \sum_{n,m}\mu^{2n}f^m E_{nm}\right]$$

Note angle dependence of damping.

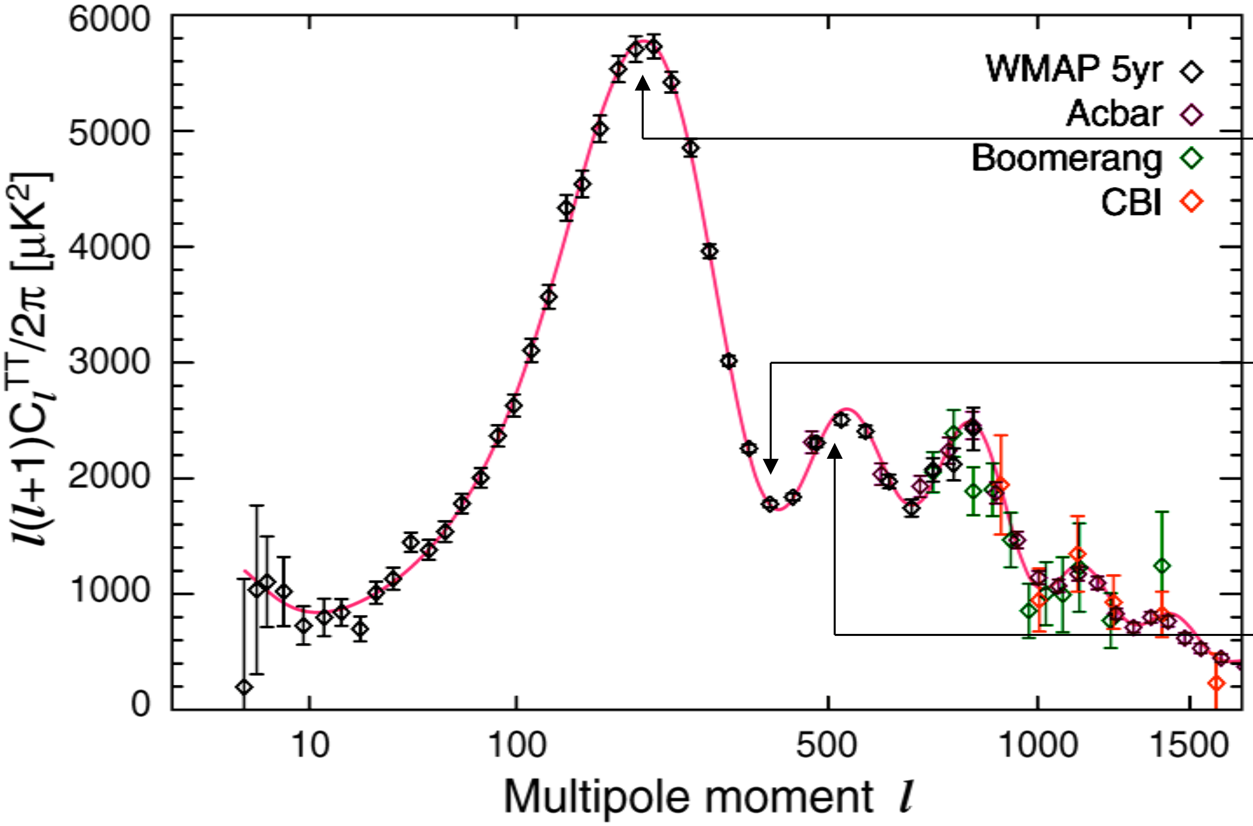
Zel'dovich damping

Mode coupling terms up to E_{44}. These terms involve b_1 and b_2.



Non-linearities and BAO

Acoustic oscillations



First "compression", at $kc_st_{ls}=\pi$. Density maxm, velocity null.

Velocity maximum

First "rarefaction" peak at $kc_st_{ls}=2\pi$

Acoustic scale is set by the *sound horizon* at last scattering: $s = c_st_{ls}$

CMB calibration

- Not coincidentally the sound horizon is extremely well determined by the structure of the acoustic peaks in the CMB.

$$\begin{aligned} s &= 146.8 \pm 1.8 \text{ Mpc} \\ &= (4.53 \pm 0.06) \times 10^{24}\text{m} \end{aligned}$$

WMAP 5th yr data

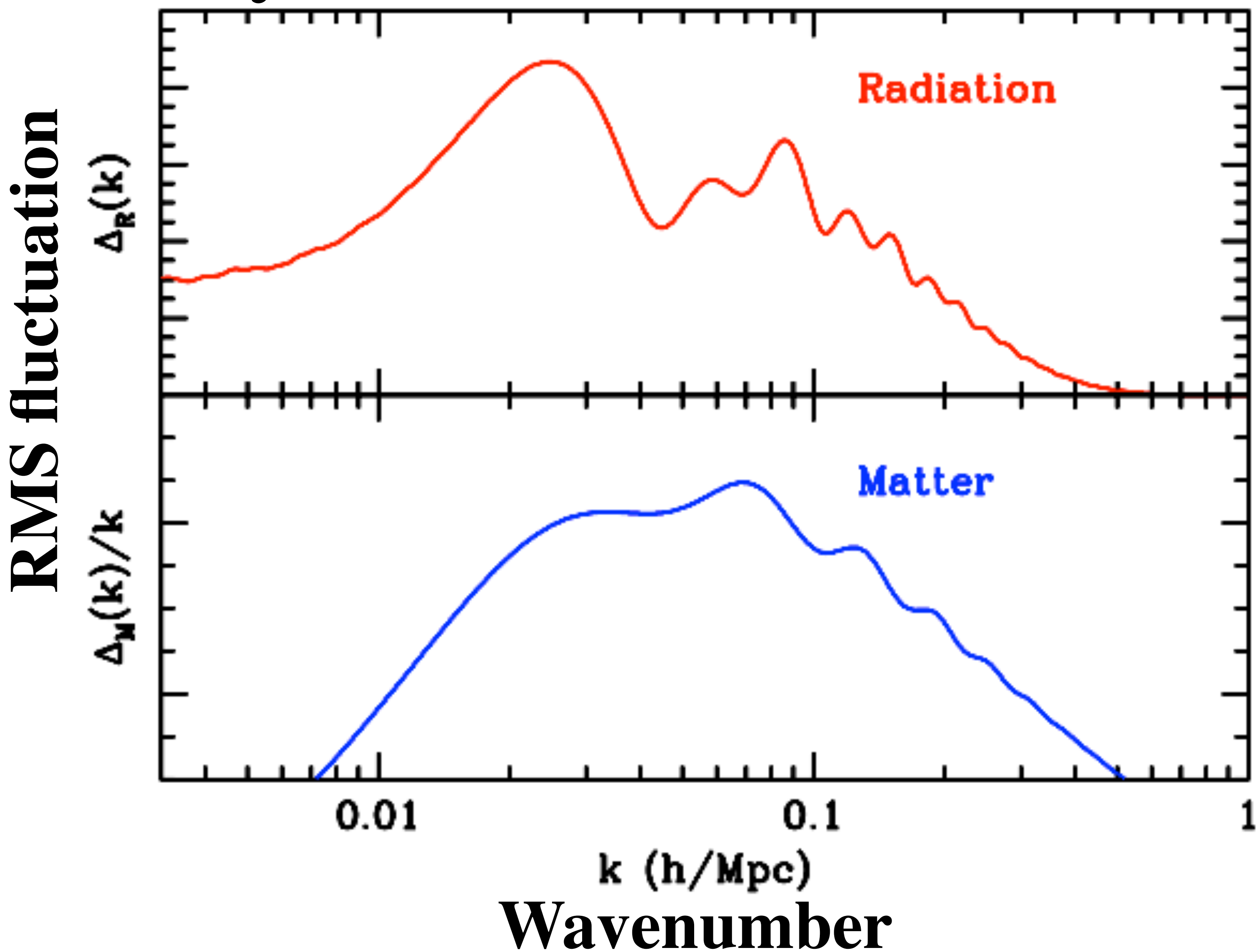
Dominated by uncertainty in ρ_m from poor constraints near 3rd peak in CMB spectrum. (Planck will nail this!)

Baryon oscillations in *P(k)*

- Since the baryons contribute ~15% of the total matter density, the total gravitational potential is affected by the acoustic oscillations with scale set by *s*.
- This leads to small oscillations in the matter power spectrum *P*(*k*).
  - No longer order unity, like in the CMB
  - Now suppressed by $\Omega_b/\Omega_m \sim 0.1$

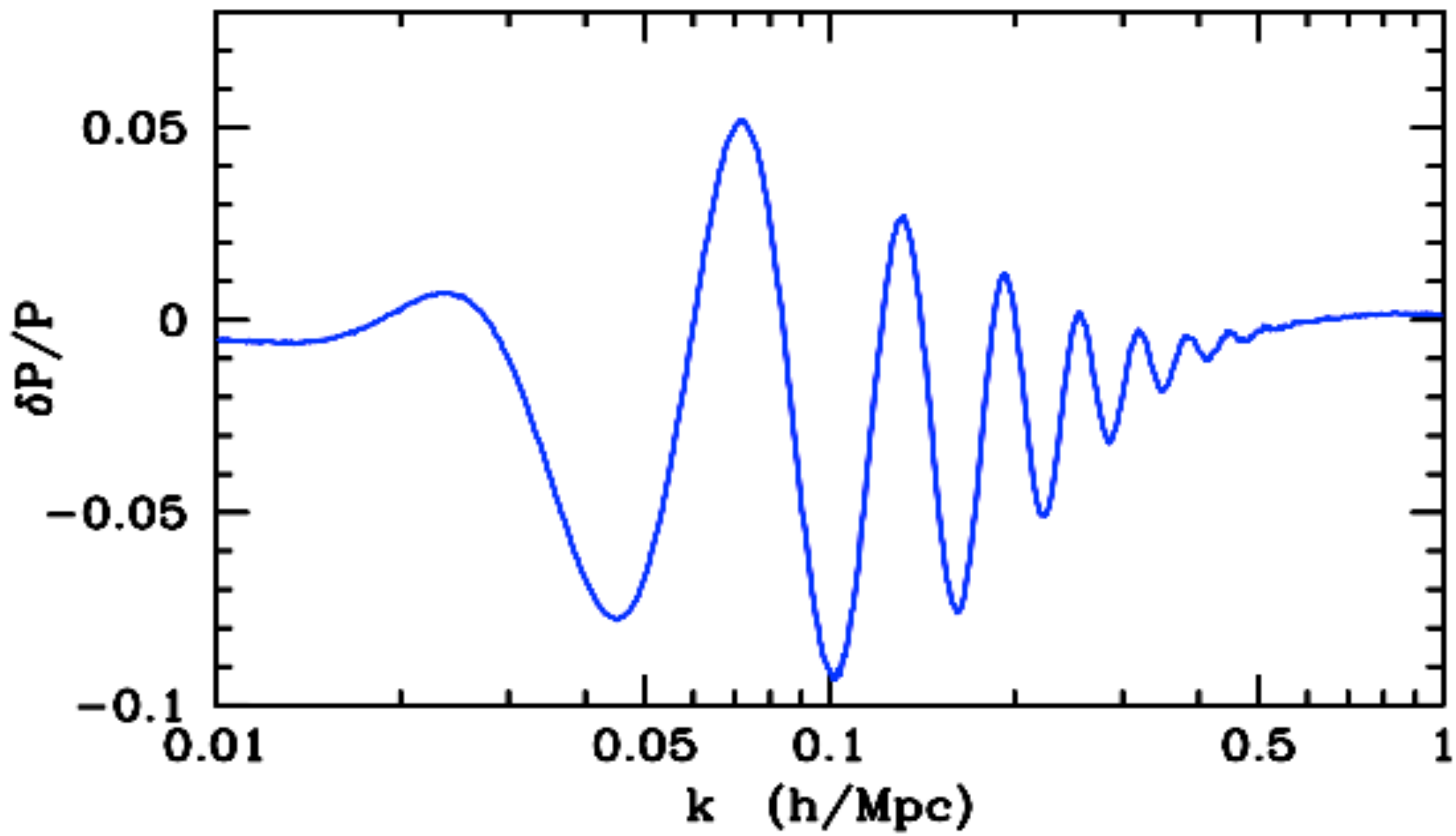
- **Note:** all of the matter sees the acoustic oscillations, not just the baryons.

Baryon (acoustic) oscillations



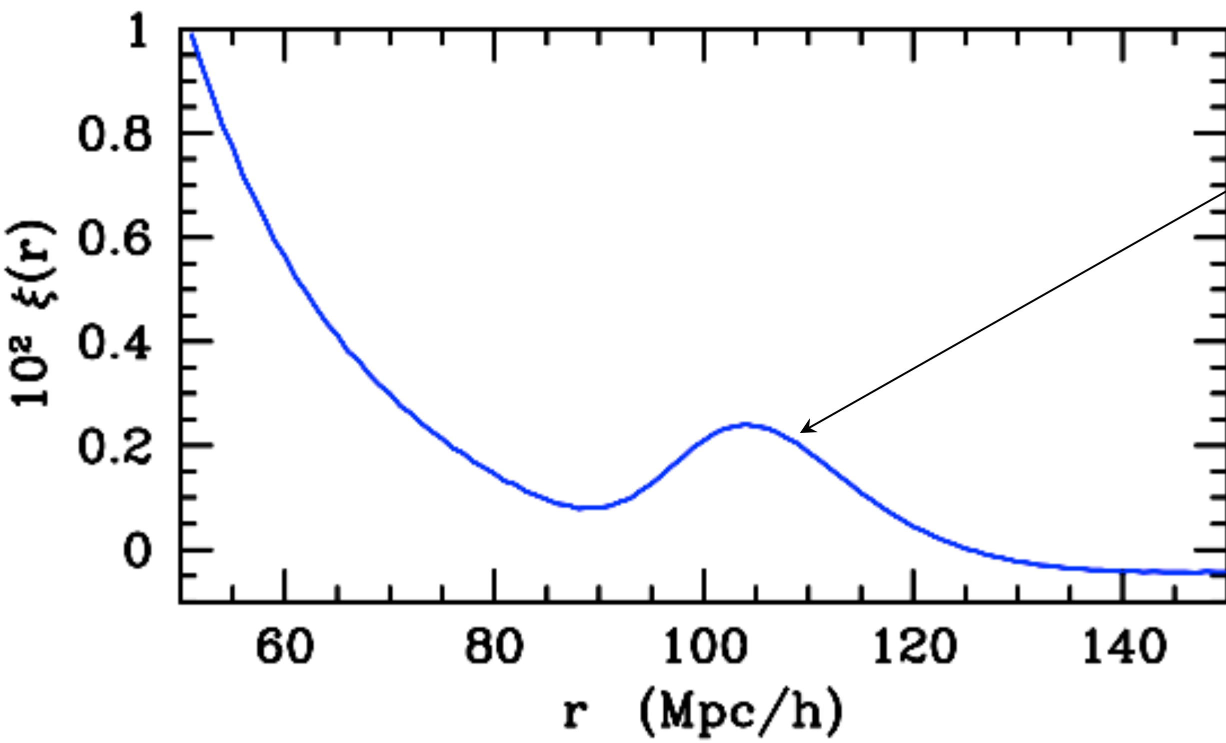
Divide out the gross trend …

A damped, almost harmonic sequence of “wiggles” in the power spectrum of the mass perturbations of amplitude *O*(10%).



In configuration space

- The configuration space picture offers some important insights.
- In configuration space we measure not power spectra but correlation functions: $\xi(r)=\int P(k)e^{ikr}d^3k=\int \Delta^2(k)j_0(kr)\, d\ln k$..
- A harmonic sequence would be a δ-function in r, the shift in frequency and diffusion damping broaden the feature.

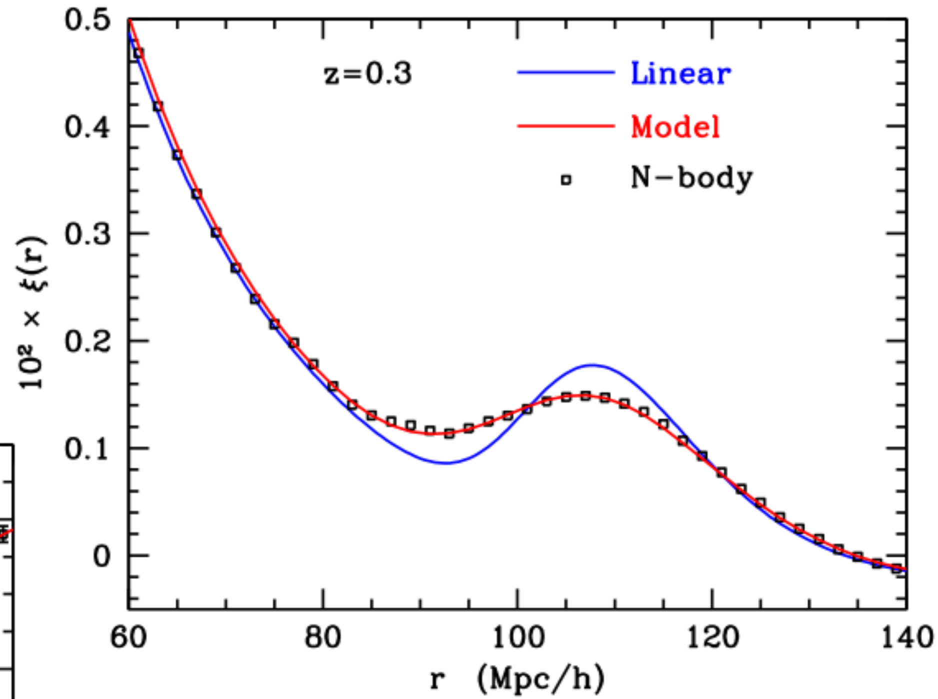


Acoustic feature at ~100 Mpc/h with width ~10Mpc/h (Silk scale)

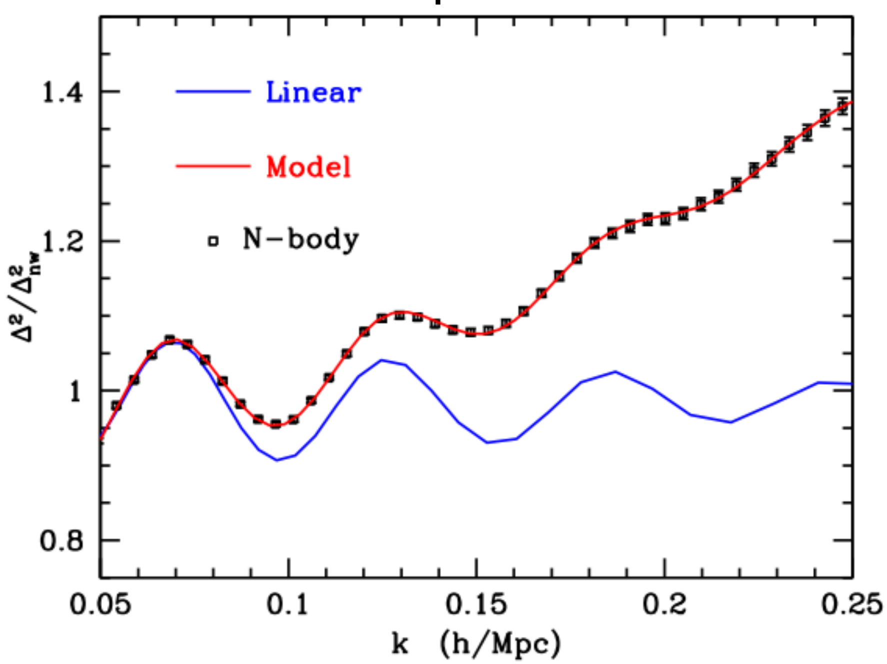
Effects of non-linearity on BAO

- Non-linear evolution has 3 effects on the power spectrum:
  - It generates “excess” high *k* power, reducing the contrast of the wiggles.
  - It damps the oscillations.
  - It generates an out-of-phase component.
- In configuration space:
  - Generates “excess” small-scale power.
  - Broadens the peak.
  - Shifts the peak.

Non-linearities smear the peak



Loss of contrast and excess power from non-linear collapse.



Broadening of feature due to Gaussian smoothing and ~0.5% shift due to mode coupling.

Understanding "shifts"

- We want to fit for the position of the acoustic feature while allowing for variations in the broadband shape (due e.g. to biasing).
  - $P_{fit}(k) = B(k)\, P_w(k,\alpha) + A(k)$
  - *B(k)* and *A(k)* are smooth functions.
    - Can take B(k)=const and A(k) as a spline, polynomial, Pade, ...
  - α measures shift relative to "fiducial" cosmology.
  - $P_w(k,\alpha)$ is a template.
    - Numerous arguments suggest $P_w(k,\alpha)=\exp[-k^2\Sigma^2/2]P_L(k/\alpha)$.
    - Take Σ *to be a free parameter, perhaps with a prior.*
- How does this do?

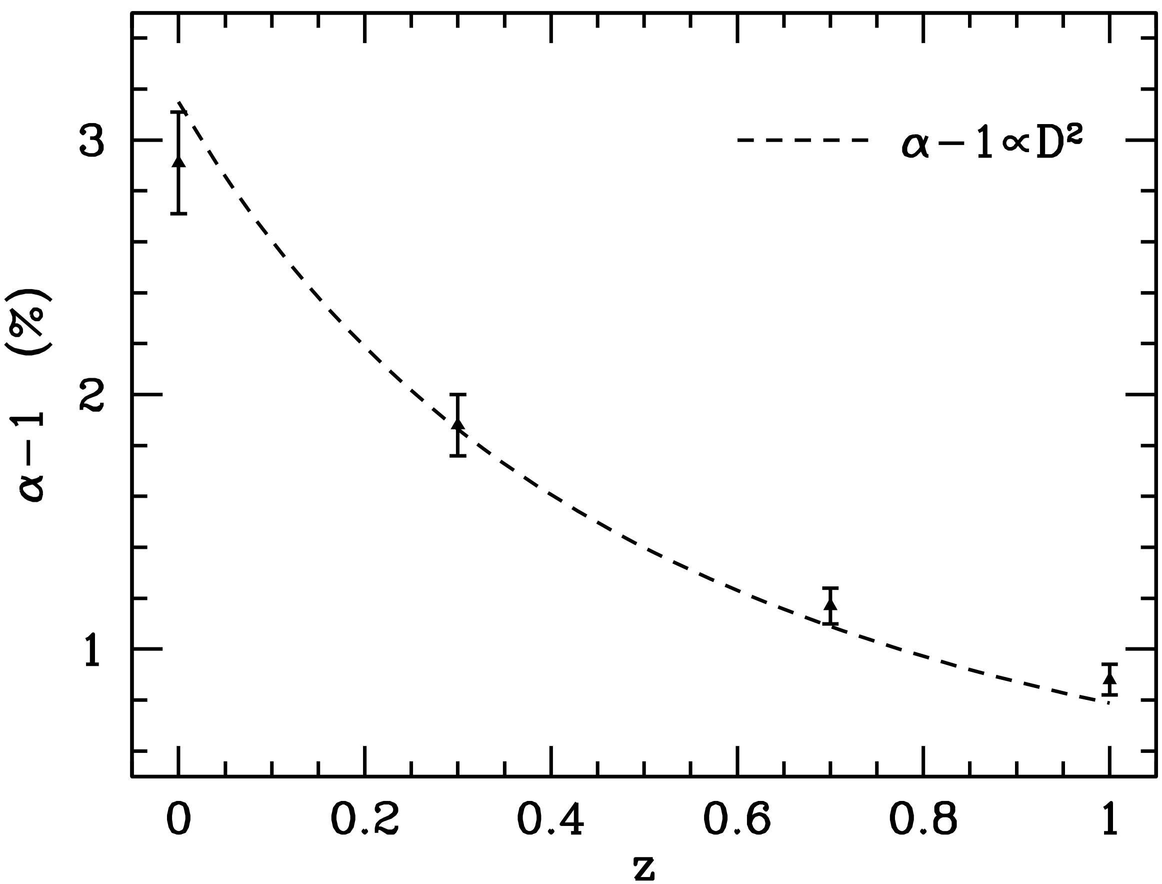
Argument from
Padmanabhan & White (2009);
see also Smith++08.

Measuring shifts in cCDM

- Any “shift” in the acoustic scale is small in ΛCDM, and therefore hard to study.
- Work with a “crazy” cosmology
  - Ω_m=1, Ω_B=0.4, h=0.5, n=1, σ_8=1.
  - Sound horizon 50h^{-1}Mpc, not 100h^{-1}Mpc.
- The fitted shifts are (α-1 in percent):

z	DM	$x\delta_L$	w/P_{22}
0.0	2.91 ± 0.20	-0.2 ±0.1	-0.03 ± 0.16
0.3	1.88 ± 0.12	-0.2 ±0.1	-0.38 ± 0.09
0.7	1.17 ± 0.07	-0.1 ±0.1	-0.12 ± 0.05
1.0	0.88 ± 0.06	-0.1 ±0.1	-0.04 ± 0.04

Shifts vs time



Amplitude of the shift vs. time (redshift) for the mass.

Shifts are consistent with D^2 scaling (dotted) suggesting an origin from 2^{nd} order terms …

Where do the shifts come from?

Recall in PT we can write $\delta=\delta^{(1)}+\delta^{(2)}+\ldots$ or
$P = \{P_{11} + P_{13} + P_{15} + \ldots\} + \{P_{22} + \ldots\} = P_{1n} + P_{mn}$.
We can isolate these two types of terms by considering the cross-spectrum of the final with the initial field, which doesn't contain P_{mn}.

z	DM	$x\delta_L$	w/P_{22}
0.0	2.91 ± 0.20	-0.2 ±0.1	-0.03 ± 0.16
0.3	1.88 ± 0.12	-0.2 ±0.1	-0.38 ± 0.09
0.7	1.17 ± 0.07	-0.1 ±0.1	-0.12 ± 0.05
1.0	0.88 ± 0.06	-0.1 ±0.1	-0.04 ± 0.04

Shifts in the cross-spectrum are an order of magnitude smaller than shifts in the auto-spectrum!

$$P_{1n}(k) \sim P_L(k) \int \prod_k \left[d^3 q_k P_L(q_k)\right] F_n(\cdots)$$

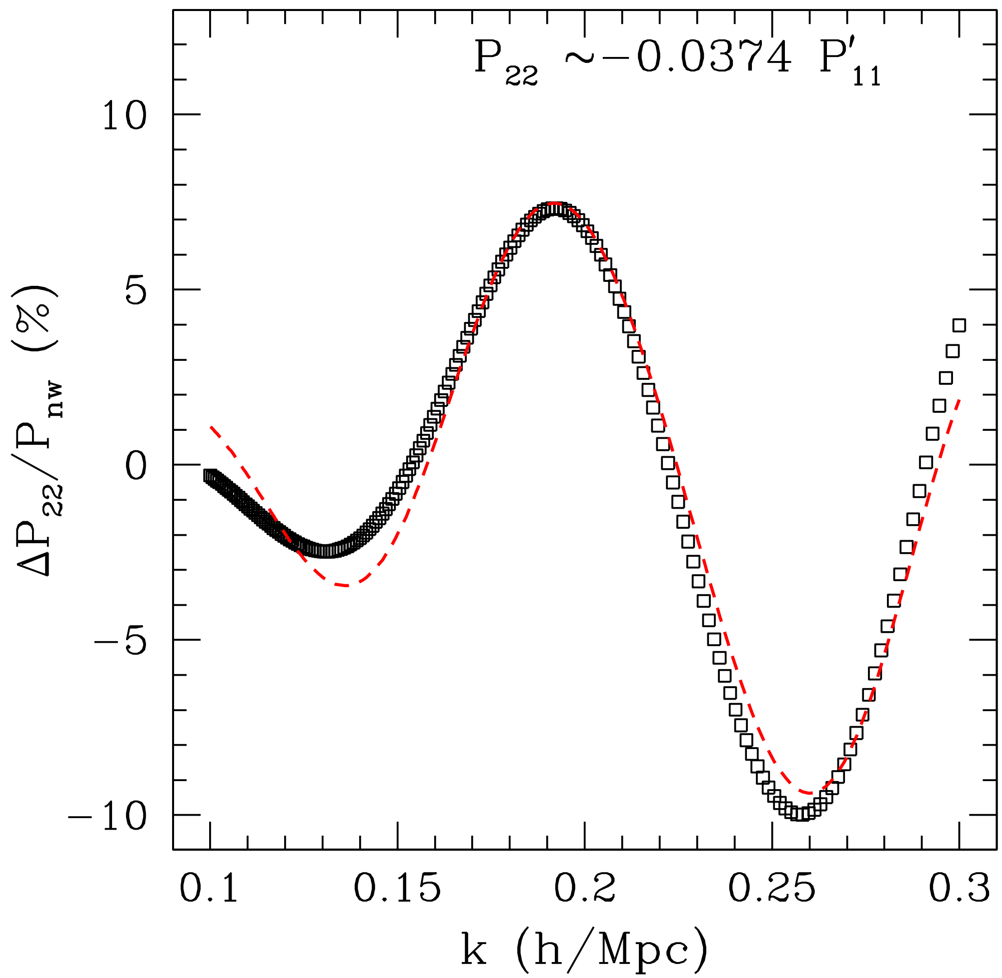
Broad kernel suppresses oscillations.

Mode-coupling terms

Recall in PT we can write $\delta=\delta^{(1)}+\delta^{(2)}+\ldots$ or
$P = \{P_{11} + P_{13} + P_{15} + \ldots\} + \{P_{22} + \ldots\} = P_{1n} + P_{mn}$.

- The P_{1n} terms are benign.
- By contrast the P_{mn} terms involve integrals of products of P_Ls times peaked kernels.
- Example: $P_{22} \sim \int P_L P_L\ F_2$ and F_2 is sharply peaked around $q_1 \approx q_2 \approx k/2$.
- Thus the $\int P_L P_L$ term contains an out-of-phase oscillation
  - $P_L \sim \ldots + \sin(kr)$: $P_L P_L F_2 \sim \sin^2(kr/2) \sim 1+\cos(kr)$
- Since cos(x)~d/dx sin(x) this gives a "shift" in the peak
  - $P(k/\alpha) \sim P(k) - (\alpha-1)\ dP/d\ln k + \ldots$

Mode-coupling approximates derivative



Up to an overall factor the mode-coupling term, P_{22}, is well approximated by $dP_L/d\ln k$.

Modified template

- This discussion suggests a modified template, which has just as many free parameters as our old template:

$$
\begin{aligned}
P_{\rm w}(k,\alpha) &= \exp\left(-\frac{k^2\Sigma^2}{2}\right) P_L(k/\alpha) \\
&+ \exp\left(-\frac{k^2\Sigma_1^2}{2}\right) P_{22}(k/\alpha)\,.
\end{aligned}
$$

- This removes most of the shift.

z	DM	$x\delta_L$	w/P_{22}
0.0	2.91 ± 0.20	-0.2 ±0.1	-0.03 ± 0.16
0.3	1.88 ± 0.12	-0.2 ±0.1	-0.38 ± 0.09
0.7	1.17 ± 0.07	-0.1 ±0.1	-0.12 ± 0.05
1.0	0.88 ± 0.06	-0.1 ±0.1	-0.04 ± 0.04

Padmanabhan & White (2009)

Biased tracers?

- In order to remove the shift we needed to know the relative amplitude of P_{11} and P_{22}.
  - For the mass, this is known.
- What do we do for biased tracers?
  - Eulerian bias

$$P_h = \left(b_1^E\right)^2 (P_{11} + P_{22}) + b_1^E b_2^E \left(\frac{3}{7}Q_8 + Q_9\right) + \frac{(b_2^E)^2}{2} Q_{13} + \cdots$$

  - Lagrangian bias

$$\begin{aligned} P_h &= \exp\left[-\frac{k^2\Sigma^2}{2}\right] \left\{ \left(1 + b_1^L\right)^2 P_{11} + P_{22} + b_1^L \left[\frac{6}{7}Q_5 + 2Q_7\right] + b_2^L \left[\frac{3}{7}Q_8 + Q_9\right] \right. \\ &+ \left. \left(b_1^L\right)^2 [Q_9 + Q_{11}] + 2b_1^L b_2^L Q_{12} + \frac{1}{2}\left(b_2^L\right)^2 Q_{13} \right\} + \cdots \end{aligned}$$

Mode-coupling integrals

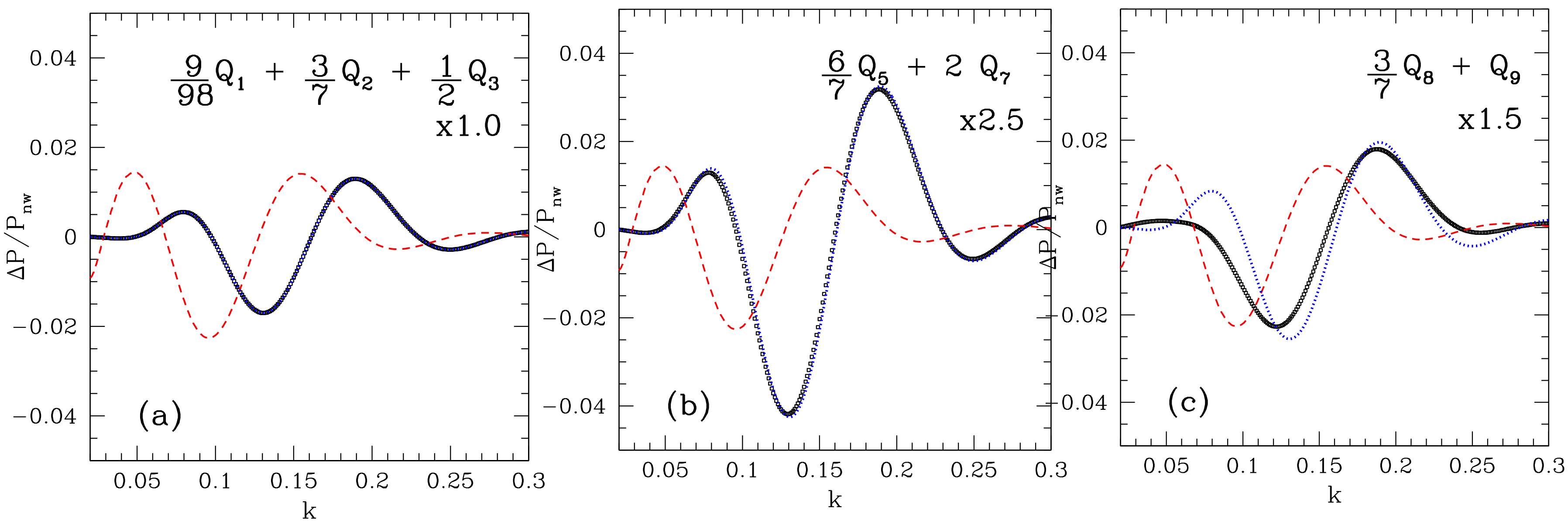
$$Q_n(k) = \frac{k^3}{4\pi^2}\int_0^\infty dr\, P_L(kr)\int_{-1}^{1} dx\, P_L(k\sqrt{1+r^2-2rx})\widetilde{Q}_n(r,x)$$

$$\begin{aligned}
\widetilde{Q}_1 &= \frac{r^2(1-x^2)^2}{y^2}, & \widetilde{Q}_2 &= \frac{(1-x^2)rx(1-rx)}{y^2},\\
\widetilde{Q}_3 &= \frac{x^2(1-rx)^2}{y^2}, & \widetilde{Q}_4 &= \frac{1-x^2}{y^2},\\
\widetilde{Q}_5 &= \frac{rx(1-x^2)}{y}, & \widetilde{Q}_6 &= \frac{(1-3rx)(1-x^2)}{y},\\
\widetilde{Q}_7 &= \frac{x^2(1-rx)}{y}, & \widetilde{Q}_8 &= \frac{r^2(1-x^2)}{y},\\
\widetilde{Q}_9 &= \frac{rx(1-rx)}{y}, & \widetilde{Q}_{10} &= 1-x^2,
\end{aligned}$$

$$\widetilde{Q}_{11} = x^2,\ \widetilde{Q}_{12} = rx,\ \widetilde{Q}_{13} = r^2$$

(Matsubara 2008)

Out-of-phase?

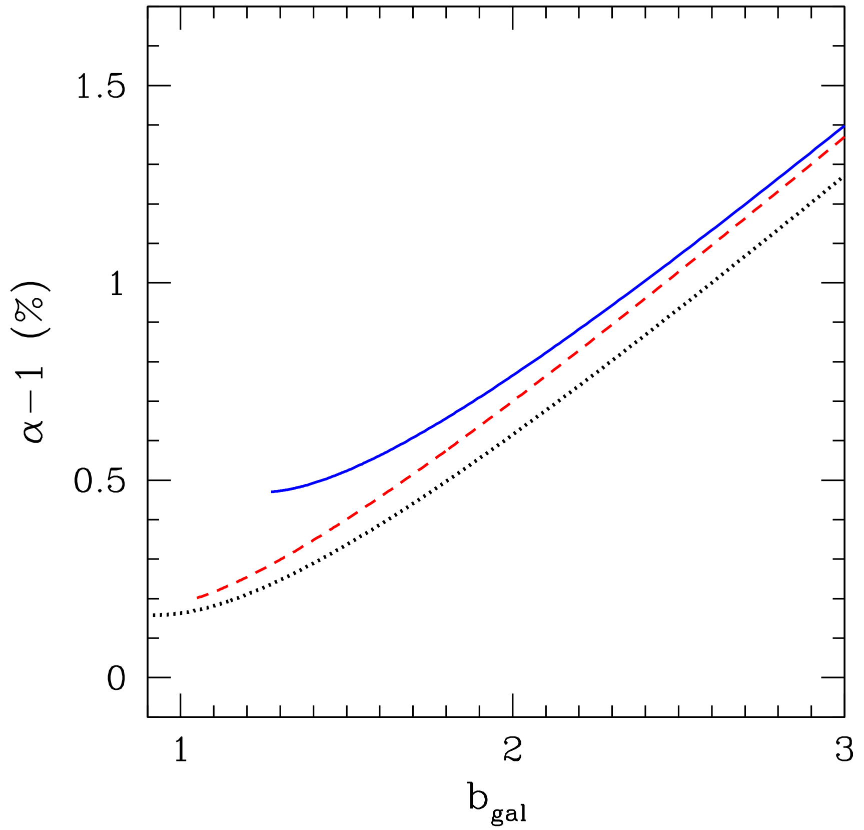


The numerous combinations that come in are also well approximated by the (log-)derivative of P_{11}! All of these terms can be effectively written as:

$$P_h = \exp\left(-\frac{k^2\Sigma^2}{2}\right)\left[\mathcal{B}_1 P_L + \mathcal{B}_2 P_{22}\right].$$

Size of the shifts?

- Simple model explains B_1-B_2 relation.
  - True for a variety of cosmologies, including ΛCDM.
  - Can also be measured from simulations (using some tricks).
- For ΛCDM the shifts are:
  - α-1~0.5% x D^2 x B_2/B_1



Shifts at z=0 for

Halos of mass M
Halos above M
N~[1+M/M_1]

At higher z the shift decreases as D^2.

Recall, the final error in BAO scale is the *uncertainty* in this correction, not the size of the correction itself!

Redshift space

- In resummed LPT we can also consider the redshift space power spectrum for biased tracers.
- For the isotropic P(k) find a similar story though now the scaling coefficients depend on $f \sim d D/d\ln a$.
  - Expressions become more complex, but the structure is unchanged.
- The amplitude of the shift increases slightly.

Perturbation theory & BAO

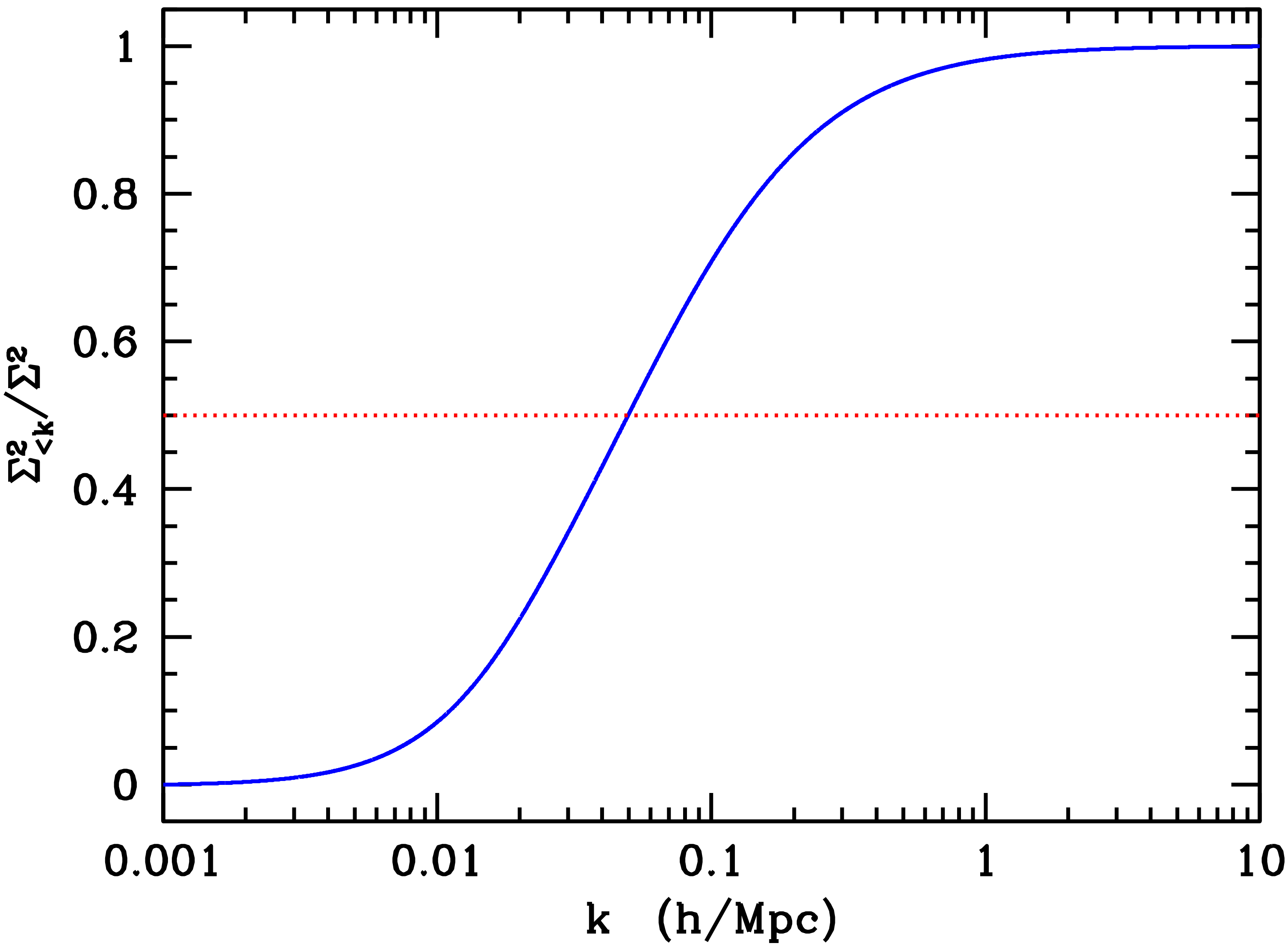
- Meiksin, White & Peacock, 1999
  - Baryonic signatures in large-scale structure (SPT)
- Crocce & Scoccimarro, 2007
  - Nonlinear Evolution of Baryon Acoustic Oscillations
- Matsubara, 2008ab
- Jeong & Komatsu, 2006, 2009
  - Perturbation theory reloaded I & II
- Pietroni, 2008; Lesgourgues et al. 2009; Anselmi et al. 2010; Elia et al. 2010
  - Flowing with time, resummation schemes.
- Padmanabhan & White 2009; Padmanabhan et al., 2009; Noh et al. 2009
  - Calibrating the baryon oscillation ruler for matter and halos
  - Reconstructing baryon oscillations: A Lagrangian theory perspective
  - Reconstructing baryon oscillations.
- Nishimichi et al., 2007, 2010; Taruya et al., 2009, 2010.
  - Characteristic scales of BAO from perturbation theory
  - Non-linear Evolution of Baryon Acoustic Oscillations from Improved Perturbation Theory in Real and Redshift Spaces

Reconstruction
an analytic understanding?

Reconstruction and LPT

- Recall that the effect of non-linearity was to broaden (and slightly shift) the acoustic peak.
- The broadening was equal to the Zel'dovich displacement.
  - Much of the broadening comes from large scales.
- Since those scales are measured by the survey, one could hope to "reconstruct" the initial, unbroadened feature.
  - Eisenstein, Seo, Sirko & Spergel (2007).
- What does this procedure do?
  - Lagrangian perturbation theory is almost perfectly suited to studying reconstruction.

Contributions to the displacement

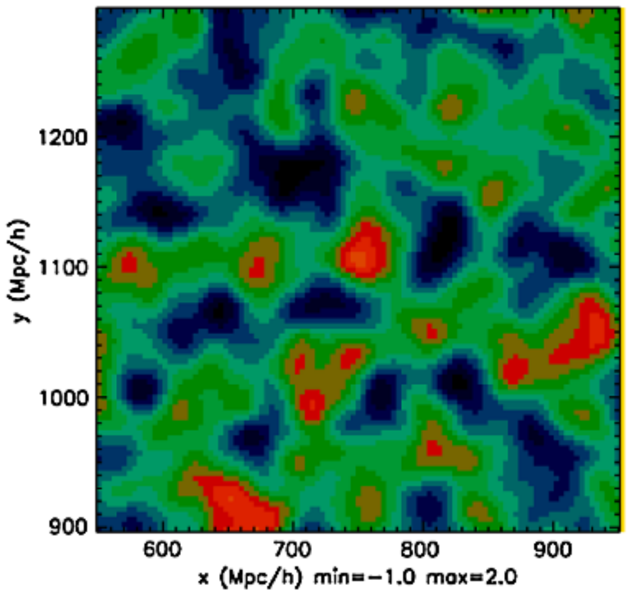


Reconstruction procedure

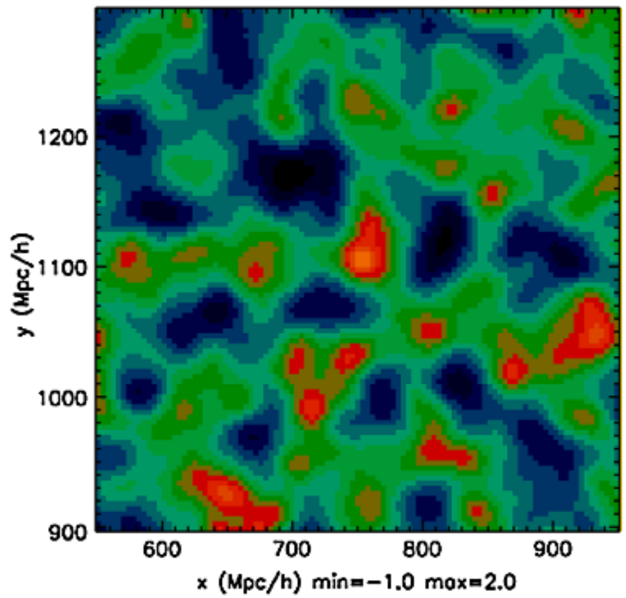
1. Smooth the density field
   - $\delta(k) \rightarrow \delta(k)\, S(k)$
2. Compute the negative Zel'dovich displacement, s, from the smooth field.
   - $\boldsymbol{s}(k) = (-i\boldsymbol{k}/k^2)\, S(k)\, \delta(k)$
3. Shift particles by s to generate "displaced" field, δ_d.
   - In linear theory $\delta_d=0$.
4. Shift spatially uniform grid of points by s to give "shifted" field, δ_s.
   - In linear theory $\delta_s=-\delta_d$.
5. Define $\delta_r=\delta_d-\delta_s$ (equals δ in linear theory).
6. Note: $S \rightarrow 0$ is equivalent to no reconstruction.

In pictures

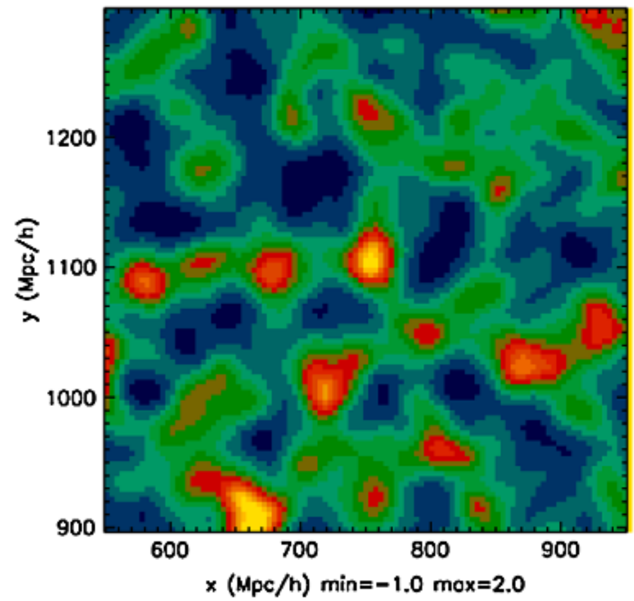
Noh++09



Initial



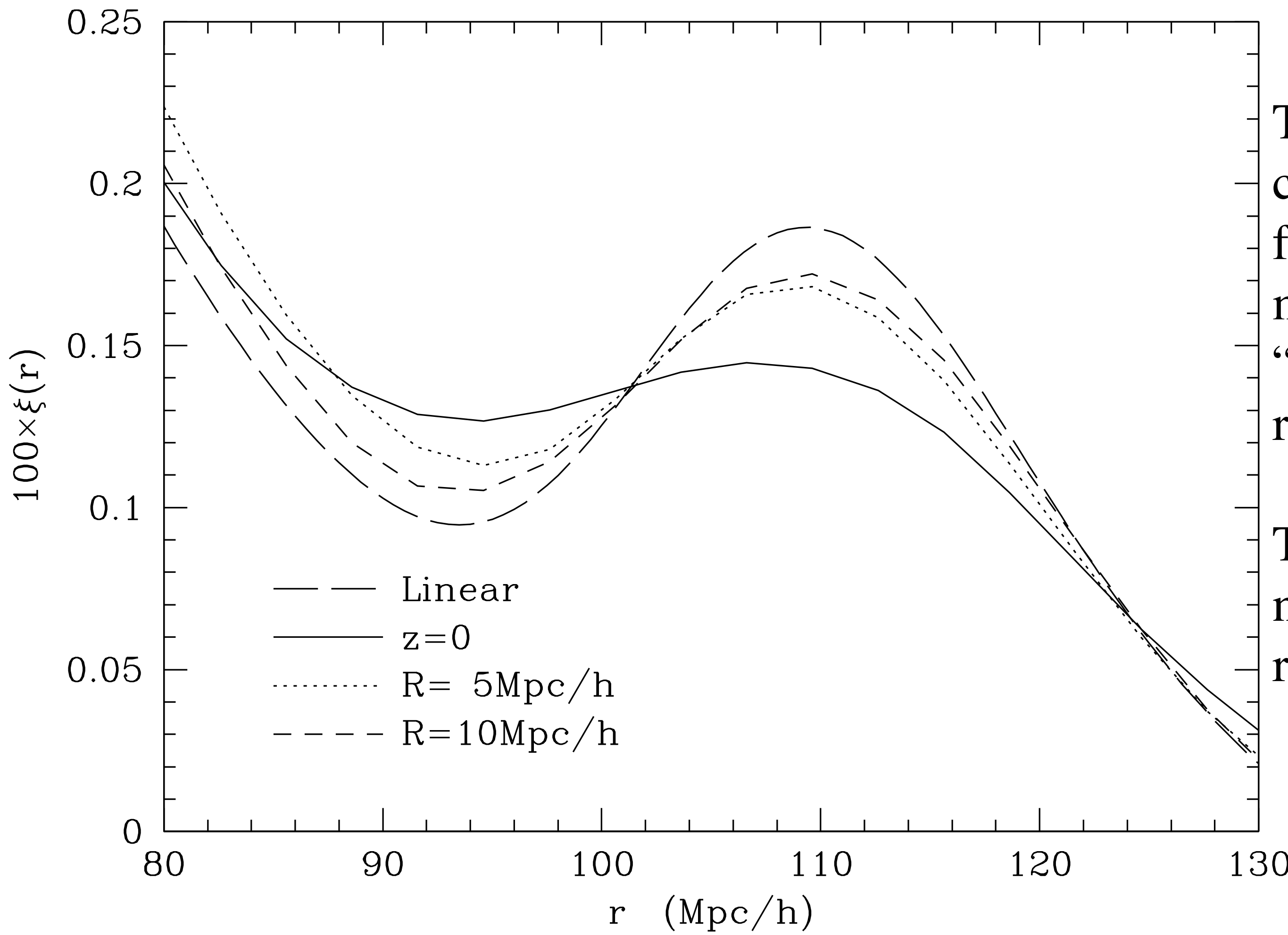
Recon



Final/NL

Note: the final field has sharper, more pronounced peaks than either the initial or reconstructed density fields.

Sharpens the peak



The z=0 correlation function of the mass in ΛCDM is "sharpened" by reconstruction.

The linear field is not fully recovered.

LPT

- Recall in LPT $$\delta(\mathbf{k}) = \int d^3q\ e^{-i\mathbf{k}\cdot\mathbf{q}} \left(e^{-i\mathbf{k}\cdot\mathbf{\Psi}(\mathbf{q})} - 1\right)$$
- The displaced field is generated by **Ψ+s**
- The shifted field is generated by **s**.
- To lowest order $\delta_r = \delta_L$.
- To next order

$$\begin{aligned}\delta_r^{(2)} &= \delta^{(2)} - \frac{1}{2}\int \frac{d^3k_1 d^3k_2}{(2\pi)^3}\delta^{(D)}\left(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}\right) \\ &\times \delta_l(\mathbf{k}_1)\delta_l(\mathbf{k}_2)\ \mathbf{k}\cdot\mathbf{L}^{(1)}(\mathbf{k}_1)\mathbf{k}\cdot\mathbf{L}^{(1)}(\mathbf{k}_2) \\ &\times [\mathcal{S}(\mathbf{k}_1) + \mathcal{S}(\mathbf{k}_2)]\end{aligned}$$

Why does reconstruction help?

A toy model

- Imagine $\Psi=\Psi_L+\Psi_H$ both Gaussian and uncorrelated.
  - Ψ_L is generated by δ_{lin},
  - Ψ_H contains no BAO.

$$P(k) = \int d^3q e^{-i\mathbf{k}\cdot\mathbf{q}} \left(\left\langle e^{-ik_i \Delta\Psi_i(\mathbf{q})} \right\rangle - 1 \right)$$

$$\left\langle e^{-i\mathbf{k}\cdot\Delta\boldsymbol{\Psi}(q)} \right\rangle = \exp\left[-\frac{1}{2} k_i k_j \left\langle \Delta\Psi_i(\mathbf{q}) \Delta\Psi_j(\mathbf{q}) \right\rangle \right]$$

$$k_i k_j \left\langle \Delta\Psi_i(\mathbf{q}) \Delta\Psi_j(\mathbf{q}) \right\rangle = 2k_i^2 \langle \Psi_i^2(\mathbf{0}) \rangle - 2k_i k_j \xi_{ij}(\mathbf{q})$$

A toy model

- $\xi_{ij}(0)=(\delta_{ij}/2)\,\Sigma^2$, and $\Sigma^2 \approx \Sigma_L{}^2$

$$P(k) = e^{-k^2\Sigma_L^2/2} \int d^3q \; e^{-ik_iq_i} \;\; e^{k_ik_j\xi_{ij}(q)} \, .$$

- Leave zero-lag piece exponentiated:

$$P_{\rm obs}(k) = e^{-\frac{1}{2}k^2\Sigma_L^2} P_L(k) + \underset{\uparrow}{P_{\rm mc}(k)} + \cdots$$

$O(\Psi_H{}^2)$ and $O(\Psi_L{}^4)$

- Now **s**(k)=-S(k)Ψ_L(k), so the displaced and shifted fields are generated by [1-S]Ψ_L+Ψ_H and –SΨ_L.

A toy model

- The reconstructed power spectrum is
  - $P_r=(\delta_s-\delta_d)^2=P_{ss}+P_{dd}-2P_{sd}$
- with:
  - $P_{ss}=\exp[-k^2\Sigma_{ss}^2/2]S^2(k)\, P_L(k)+\ldots$
  - $P_{dd}=\exp[-k^2\Sigma_{dd}^2/2][1-S(k)\,]^2P_L(k)+\ldots$
  - etc.
- And modified damping terms (e.g.):

$$\Sigma_{ss}^2 = \frac{1}{3\pi^2}\int dp\ \mathcal{S}^2(p)P_L(p)$$

- The effect of the S and [1-S] terms and the structure of the damping is to “effectively” reduce Σ to ~0.5 Σ.

LPT

- A very similar calculation carries through in the full LPT, except you have to keep more terms in the exponential if things aren't all Gaussian.
- The damping turns out to be the same.
  - We were working to lowest order in Σ, so this is not surprising.
- You additionally get the mode-coupling terms.
  - Slightly painful since you need to redo all of Matsubara with 3 different spectra.
- Find that the mode-coupling term is suppressed.

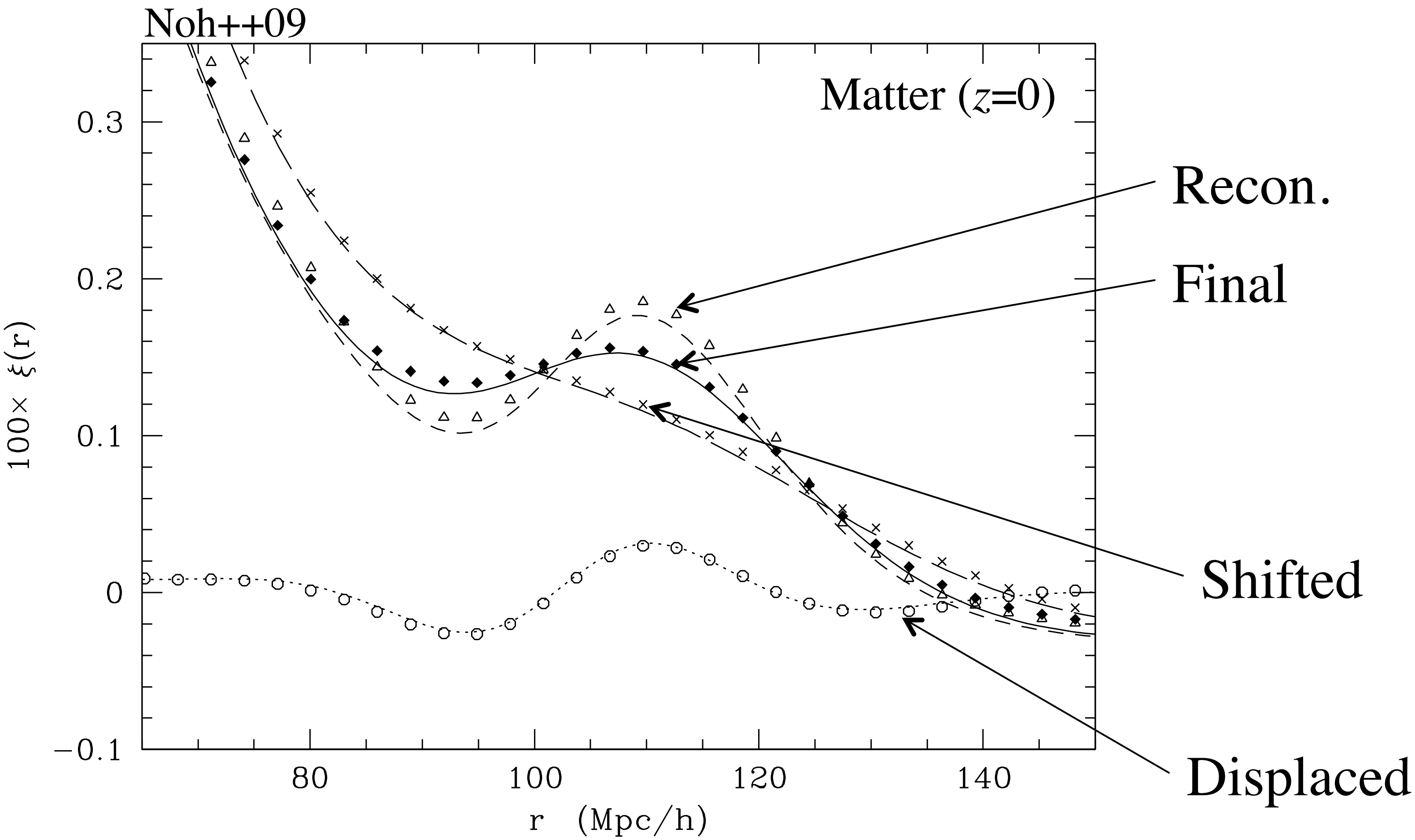
The details

$$
\begin{aligned}
P^{dd} \quad \propto \quad & P_L\bar{\mathcal{S}}^2 + \frac{9}{98}Q_1 + \frac{3}{7}Q_2^{(1d1d)} + \frac{1}{2}Q_3^{(dddd)} \\
+ \quad & \bar{\mathcal{S}}\left[\frac{10}{21}R_1 + \frac{6}{7}R_2^{(d)}\right] \\
+ \quad & \langle F'\rangle \left[2P_L\bar{\mathcal{S}} + \frac{6}{7}Q_5^{(1d11)} + \boxed{2Q_7^{(1ddd)}} + \right. \\
& \left. \frac{10}{21}R_1 + \frac{6}{7}R_2^{(d)} + \frac{6}{7}\bar{\mathcal{S}}(R_1 + R_2)\right] \\
+ \quad & \langle F''\rangle \left[\frac{3}{7}Q_8 + Q_9^{(1d1d)}\right] \\
+ \quad & \langle F'\rangle^2 \left[P_L + \frac{6}{7}(R_1 + R_2) + Q_9^{(1d1d)} + Q_{11}^{(11dd)}\right] \\
+ \quad & 2\langle F'\rangle\langle F''\rangle Q_{12}^{(111d)} + \frac{1}{2}\langle F''\rangle^2 Q_{13} \qquad (1)
\end{aligned}
$$

Noh++09; based on Matsubara 07 & 08

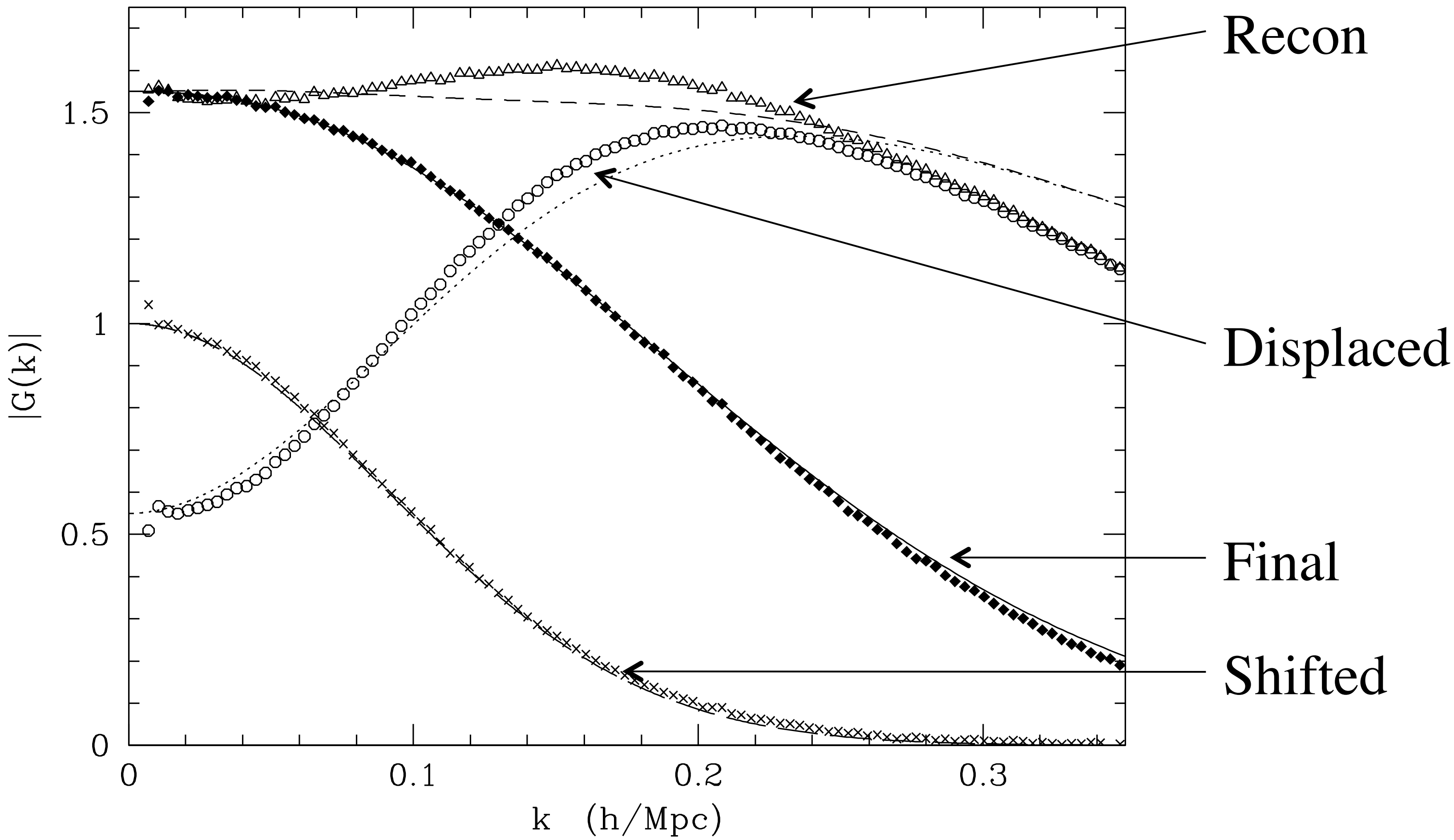
$$Q_7^{(1ddd)}(k) = \frac{k^3}{(2\pi)^2}\int_0^\infty dr\ P_L(kr)\bar{\mathcal{S}}(kr)\int_{-1}^{+1} d\mu\ P_L(ky)\bar{\mathcal{S}}(ky)\bar{\mathcal{S}}(ky)\widetilde{Q}_7(r,\mu)$$

LPT agrees with simulations



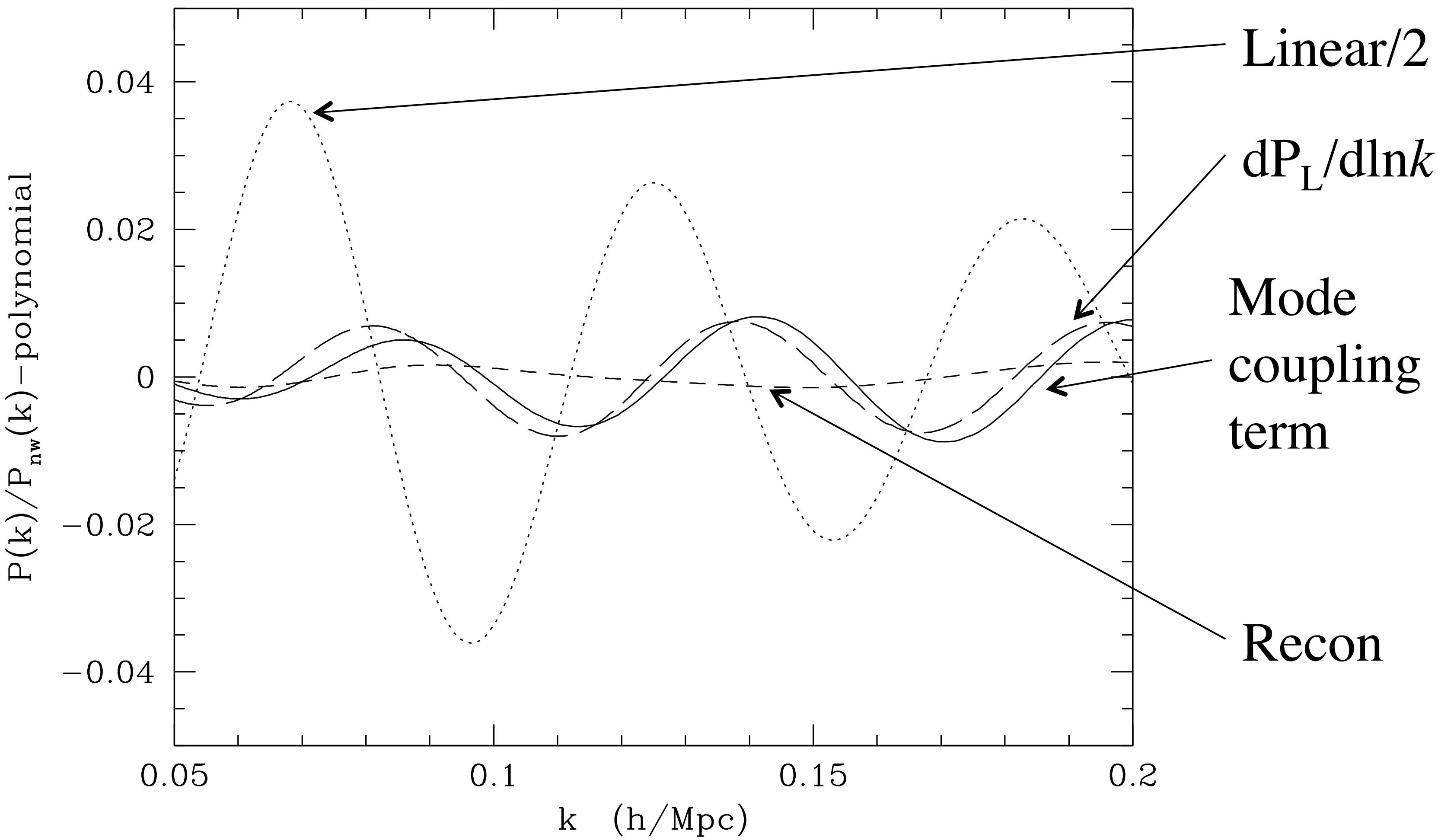
Coherence regained

Noh++09



The cross-correlation between the initial field and the other fields for halos above 10^{13}.

Out-of-phase term reduced



Out-of-phase terms in P(k) for halos more massive than 10^{13}.

Effects of shot-noise

- Within the LPT formalism the effects of shot-noise from finite galaxy number density are easy to include.
- The largest effect is a change in the damping scale:

$$\Sigma_{ss}^2 \quad \rightarrow \quad \frac{1}{3\pi^2}\int dp\ \mathcal{S}^2(p)\left[P_L(p) + P_N(p)\right]$$

$$\Sigma_{dd}^2 \quad \rightarrow \quad \frac{1}{3\pi^2}\int dp\ \left[1 - \mathcal{S}(p)\right]^2 P_L(p) + \mathcal{S}^2(p)P_N(p)\,,$$

- where $P_N=1/(b^2 n)$ is the shot-noise power.
- Gains saturate around $n \sim 10^{-4}$ $(h/\mathrm{Mpc})^3$.

White (2010)

“Strong” non-linearity

Martin White
UCB/LBNL

Limited options

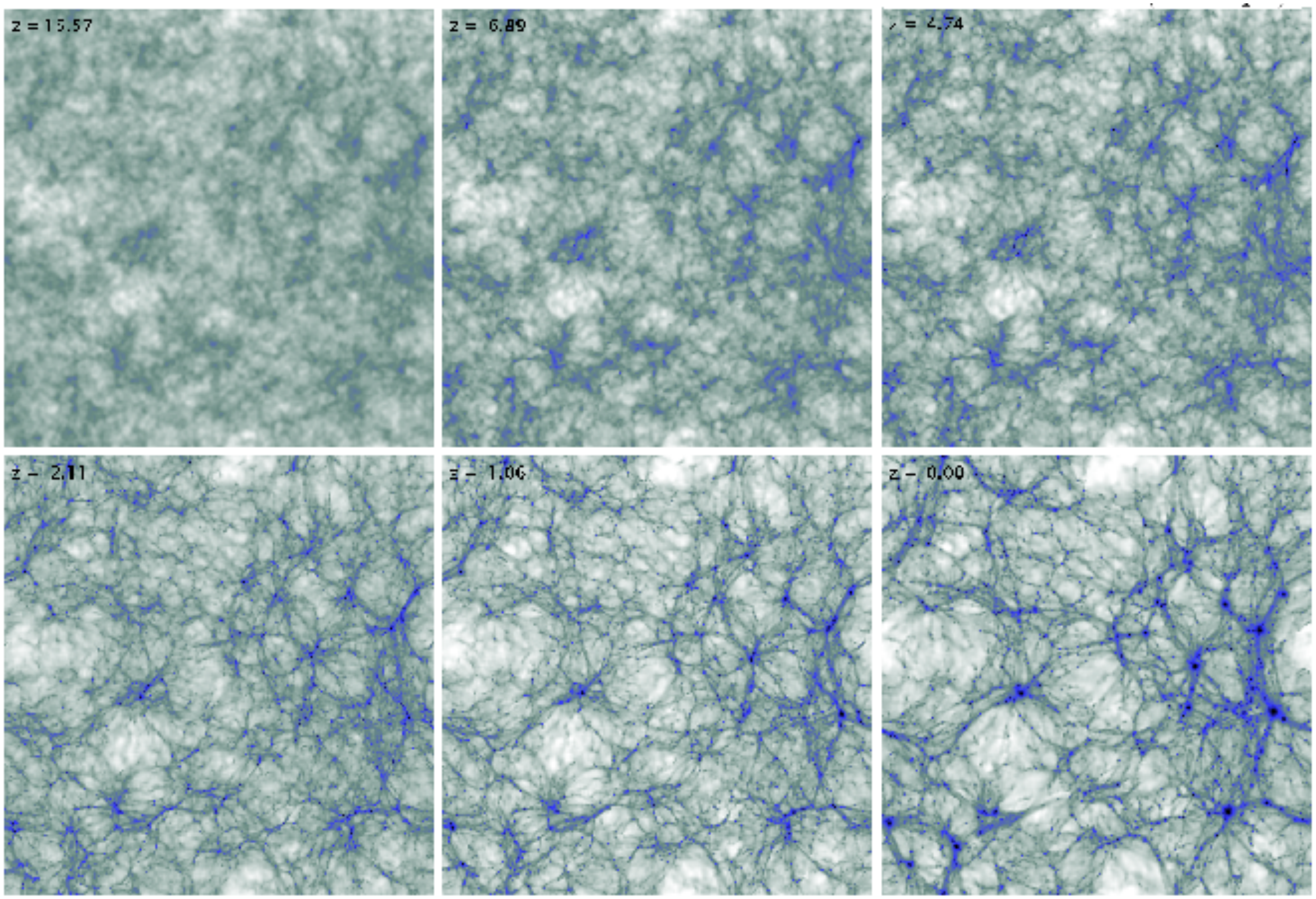
- Beyond a certain scale, linear perturbation theory breaks down
  - Definition of “non-linear scale”?
- At this point we have few options:
  - Analytical models of non-linear growth.
    - Zel’dovich approximation.
    - Spherical top-hat collapse.
  - Perturbation theory.
    - Realm of validity? Convergence criterion?
    - Good for small corrections to almost linear problems.
  - Direct simulation.
    - Numerical convergence.
    - What models to run?
    - Missing physics.

Zel'dovich approximation

- Assume particles move in a straight line with their linear perturbation theory velocity.
- Defines a mapping from initial (Lagrangian) position, q, to final (Eulerian) position, x:
  - $x=q+\Psi$ with $\Psi(q,t)=D(t)\Psi(q)$ and $\Psi_i=d\Phi/dq_i$
  - $\Psi_k = -ik/k^2\ \delta_k$
- If the initial field is uniform, the final density is the Jacobian of this mapping.
  - $\rho\sim[(1-D\alpha)(1-D\beta)(1-D\gamma)]^{-1}$
  - α,β,γ e-values of $-d^2\Phi/dq_idq_j$
- Collapse takes place first along largest e-value (pancake/sheet), then middle (filament) then final (halo).

The cosmic web

The Zel’dovich approximation, plus the statistics of Gaussian fields, qualitatively describes large-scale structure.



Springel, Hernquist & White (2000)

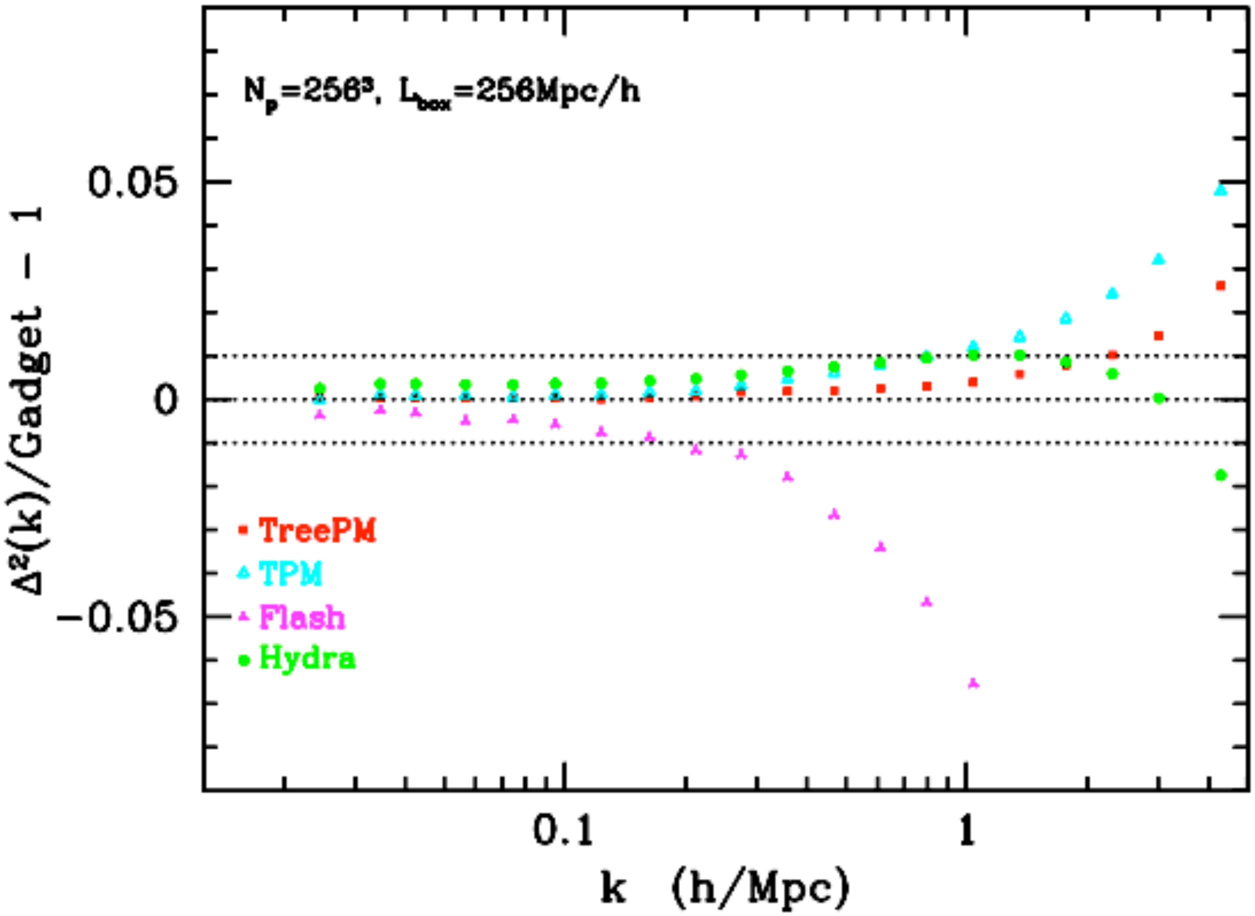
Numerical simulations

- Our ability to simulate structure formation has increased tremendously in the last decade.
- Direct simulation of the N-body problem
  - Begin at early times, but during matter domination, by displacing particles from an initial grid using 1LPT or 2LPT.
  - Monte-Carlo integration of the Vlasov equation using “super-particles” which move along the characteristics.
  - Soften the forces to avoid particle-particle scattering or integrating unphysical, tight, orbiting particles.
  - Want to approach the “fluid” limit with very large N.
  - Pure N-body codes scale “almost” perfectly.
- Our understanding of -- or at least our ability to describe -- galaxy formation has also increased dramatically.
  - Most cosmology probes observe galaxies.
  - The fundamental unit of structure theoretically is the dark matter halo.
  - Galaxies live in dark matter halos in ways we increasingly understand.

Numerical convergence

- Numerous tests of numerical convergence can be found in:
  - Heitmann et al. (2010; ApJ, 715, 104)
  - Heitmann et al. (2010; ApJ, 705, 156)
- Need to worry about
  - Starting redshift and method.
  - Force accuracy and softening.
  - Time stepping.
  - Box size.
  - Number of particles.
  - Method of computing statistic from particles.
  - How to choose *which* cosmologies to run.

Accuracy - currently demonstrated



Updated from Heitmann et al. (2007)

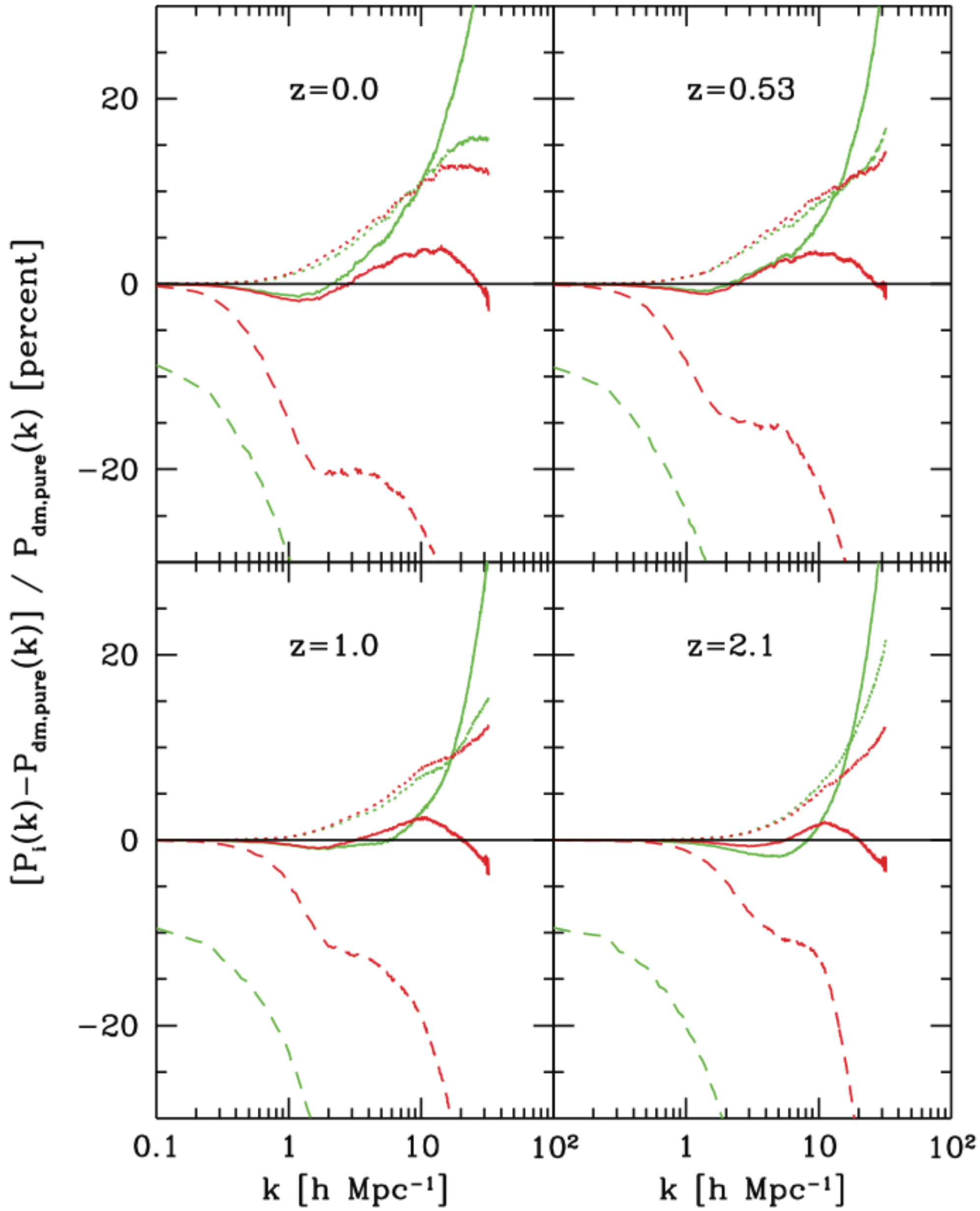
Only a sub-sample of the codes are shown here.

All codes started from the same ICs and analyzed with the same *P(k)* codes.

Extra physics

- As we go to smaller scales, we must go beyond the “pure” N-body problem and include additional physics.
  – Hydrodynamics solvers well developed.
  – Gas cools dramatically in deep potential wells, reaching high densities in a clumpy, multiphase, turbulent, magnetized ISM where it can form stars, which give off winds and radiation and go supernova injecting momentum and energy into the surrounds and have active galactic nuclei which can impart energy to their enviroments, …
- There is little scale separation between including “gas” physics and including star formation, feedback, etc. so results typically depend on sub-grid models.

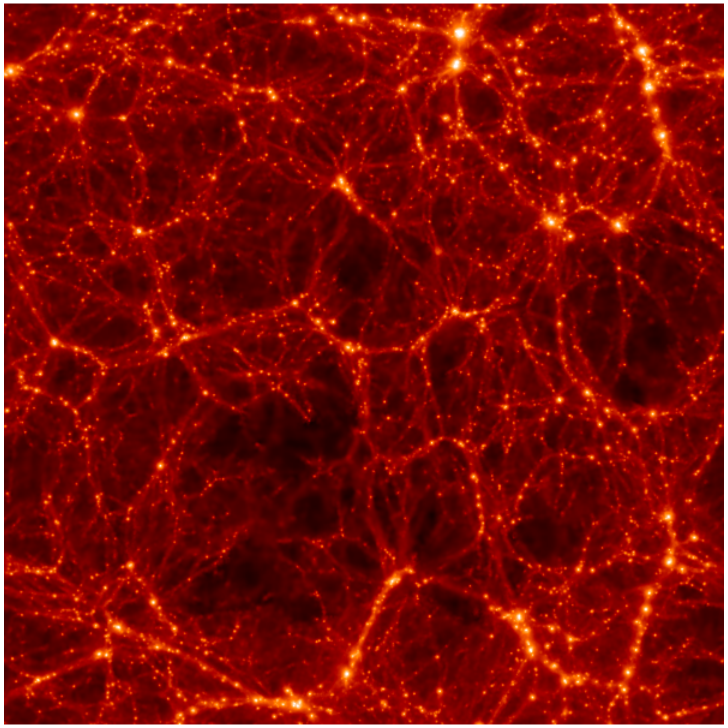
An example



One possibility, from Jing et al. (2006), for the effects of baryons (red) and baryons including star-formation and feedback (green) on the total matter (solid), dark matter (dotted) and gas (dashed).

Characteristics of LSS

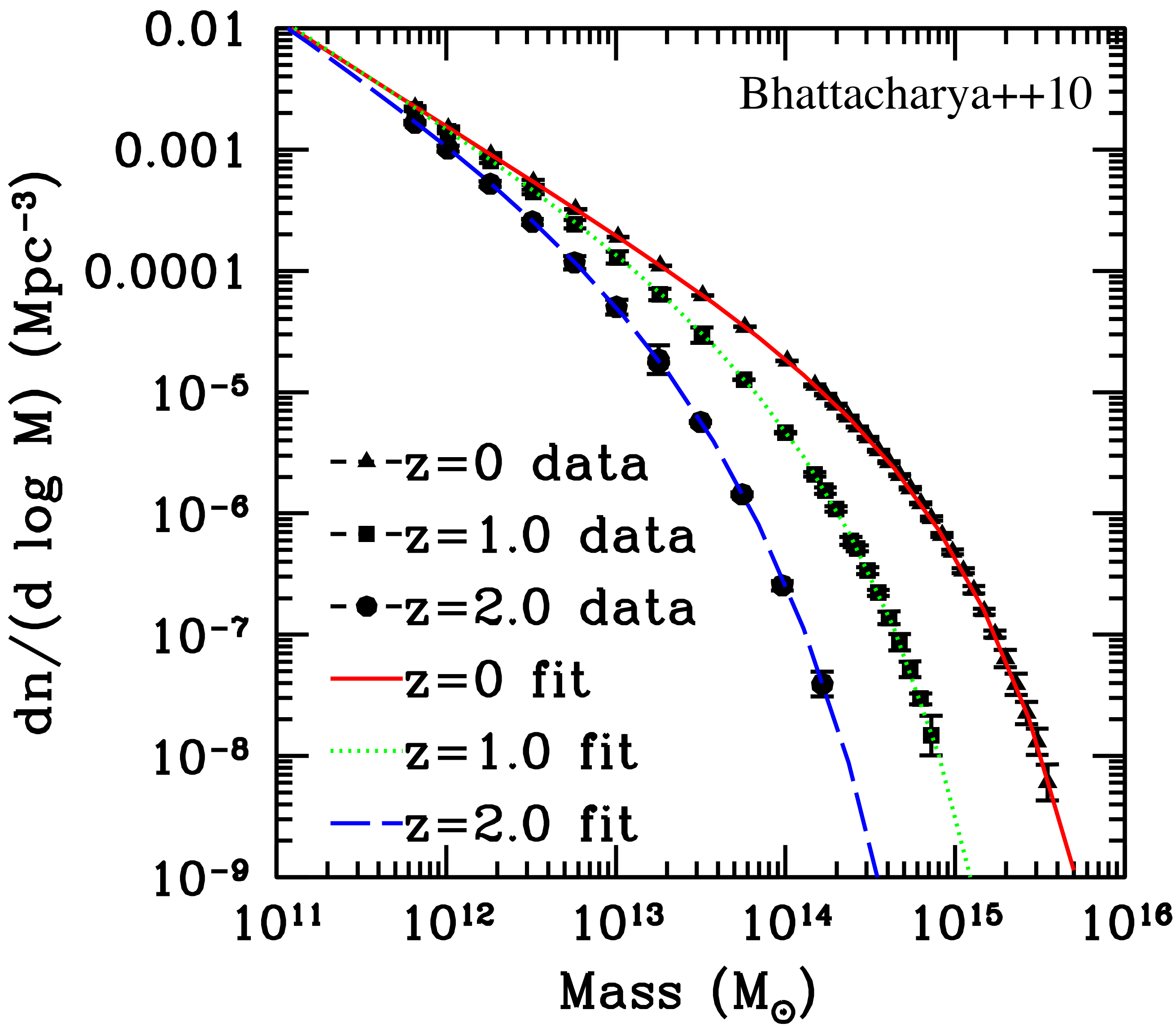
- Large-scale structure forms a beaded filamentary web of dark matter halos.
  - Number of halos vs. mass (etc.).
  - Spatial distribution of halos (vs. ?).
  - Properties of DM halos.
  - Beyond DM.



Halo abundance

- Almost all of the mass resides in (approximately) virialized halos.
- Space density of halos depends primarily (exclusively?) on mass.
- There are a large number of low mass halos and few high mass halos.
  - Very roughly $dn \sim m^{-2}\, e^{-m}$
  - As time proceeds the "characteristic" mass scale increases.
- The mass function is almost cosmology independent (in scaled units).
  - This universality is not fully understood.
- Mass functions are used in many applications in cosmology.

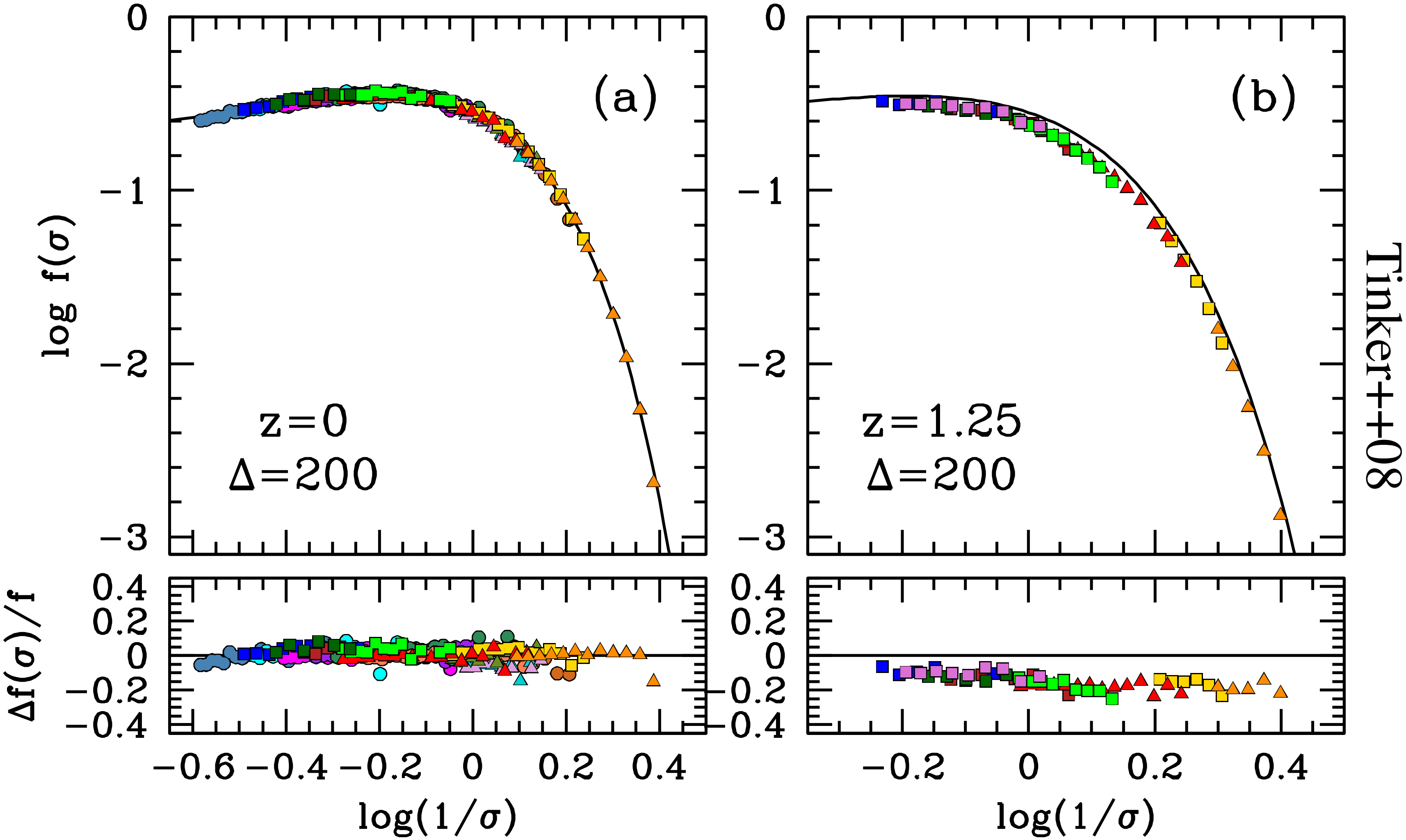
Mass function



Note the dynamic range in this figure!

Halo abundance: scaled units

$$\frac{dn}{dM} = f(\sigma)\ \frac{\bar{\rho}}{M}\ \frac{d\ln\sigma^{-1}}{dM}$$



Other fitting forms

(A detailed study of universality and numerical issues can be found in Bhattacharya++10 from which this table is taken)

MASS FUNCTION FITTING FORMULAE DERIVED IN PREVIOUS STUDIES

Reference	Fitting function f(σ)	Mass Range	Redshift range
Sheth & Tormen (2002)	$f_{ST}(\sigma) = 0.3222\sqrt{\frac{2(0.75)}{\pi}}\exp\left[-\frac{0.75\delta_c^2}{2\sigma^2}\right]\left[1+\left(\frac{\sigma^2}{0.75\delta_c^2}\right)^{0.3}\right]\frac{\delta_c}{\sigma}$	Unspecified	Unspecified
Jenkins et al. (2001)	$0.315\exp\left[-\lvert\ln\sigma^{-1}+0.61\rvert^{3.8}\right]$	$-1.2 \leq \ln\sigma^{-1} \geq 1.05$	z= 0-5
Warren et al. (2006)	$0.7234\left(\sigma^{-1.625}+0.2538\right)\exp\left[-\frac{1.1982}{\sigma^2}\right]$	$(10^{10}-10^{15})$ $h^{-1}M_\odot$	z=0
Reed et al. (2007)	$0.3222\sqrt{\frac{2(0.707)}{\pi}}\left[1+\left(\frac{\sigma^2}{0.707\delta_c^2}\right)^{0.3}+0.6G_1(\sigma)+0.4G_2(\sigma)\right]$ $\times\frac{\delta_c}{\sigma}\exp\left[-\frac{0.764\delta_c^2}{2\sigma^2}-\frac{0.03}{(n_{eff}+3)^2(\delta_c/\sigma)^{0.6}}\right]$	$-0.5 \leq \ln\sigma^{-1} \geq 1.2$	z=0-30
Manera et al. (2010)	$f_{ST}(\sigma) = 0.3222\sqrt{\frac{2a}{\pi}}\exp\left[-\frac{a\delta_c^2}{2\sigma^2}\right]\left[1+\left(\frac{\sigma^2}{a\delta_c^2}\right)^{p}\right]\frac{\delta_c}{\sigma}$	$(3.3\times10^{13}-3.3\times10^{15})$ $h^{-1}M_\odot$	z=0-0.5
Crocce et al. (2010)	$A(z)\left[\sigma^{-a(z)}+b(z)\right]\exp\left[-\frac{c(z)}{\sigma^2}\right]$	$(10^{10}-10^{15})$ $h^{-1}M_\odot$	z=0-1

$$f(\sigma) = \frac{M}{\bar{\rho}}\ \frac{dn}{d\ln\sigma^{-1}} \quad , \quad \int_0^\infty d\ln\sigma\ f(\sigma) = 1$$

Excursion set theory vs. peaks

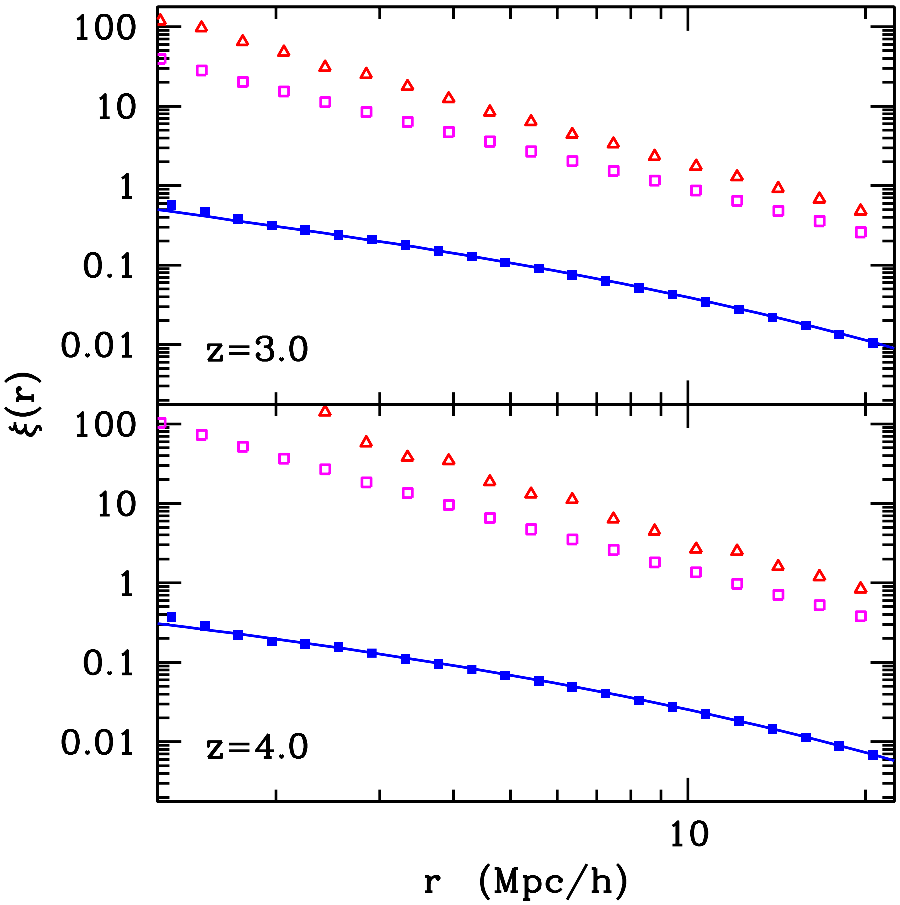
- Excursion set formalism
  - The most popular "theory".
  - The fraction of mass in halos more massive than *M* is related to the fraction of volume in which the smoothed initial density field is above some threshold, δ_c.
  - Mass function related to random walk.
    - Press-Schechter 1974; Bond, Cole, Efstathiou & Kaiser 1991.
  - Spherical collapse vs. elliptical collapse approx.
    - Mo & White, Sheth & Tormen, Zhang & Lam, …
  - How to deal with "non-locality" of halo collapse.
- Statistics of (Gaussian) peaks plus a model for halo collapse (spherical or ellipsoidal).
    - Bardeen, Bond, Kaiser & Szalay 1986
      - Based on Rice (1944; 1945) who studied 1D Gaussian fields as models of noise in communications devices.
    - Bond & Myers 1996.
    - Dalal, Lithwick & White 201X.

Excursion set theory vs. peaks

- Allow computation of mass function from statistics of initial field.
  - Choose a filter shape, and compute integrals of linear theory power spectrum and plug in formulae.
    - Not all methods self-consistent.
  - Reasonable success for mass function often improved by adjusting formulae to “fit” N-body simulations.
  - Less success for conditional mass function, merger rates etc.
  - Beware when extrapolating!

Halo bias

- The clustering of the rare, massive dark matter halos is enhanced relative to the general mass distribution
  - Kaiser 1984; Efstathiou++88; Cole & Kaiser 1989; Bond++91; Mo & White 1996; Sheth & Tormen 1999; …; Tinker++10; ...

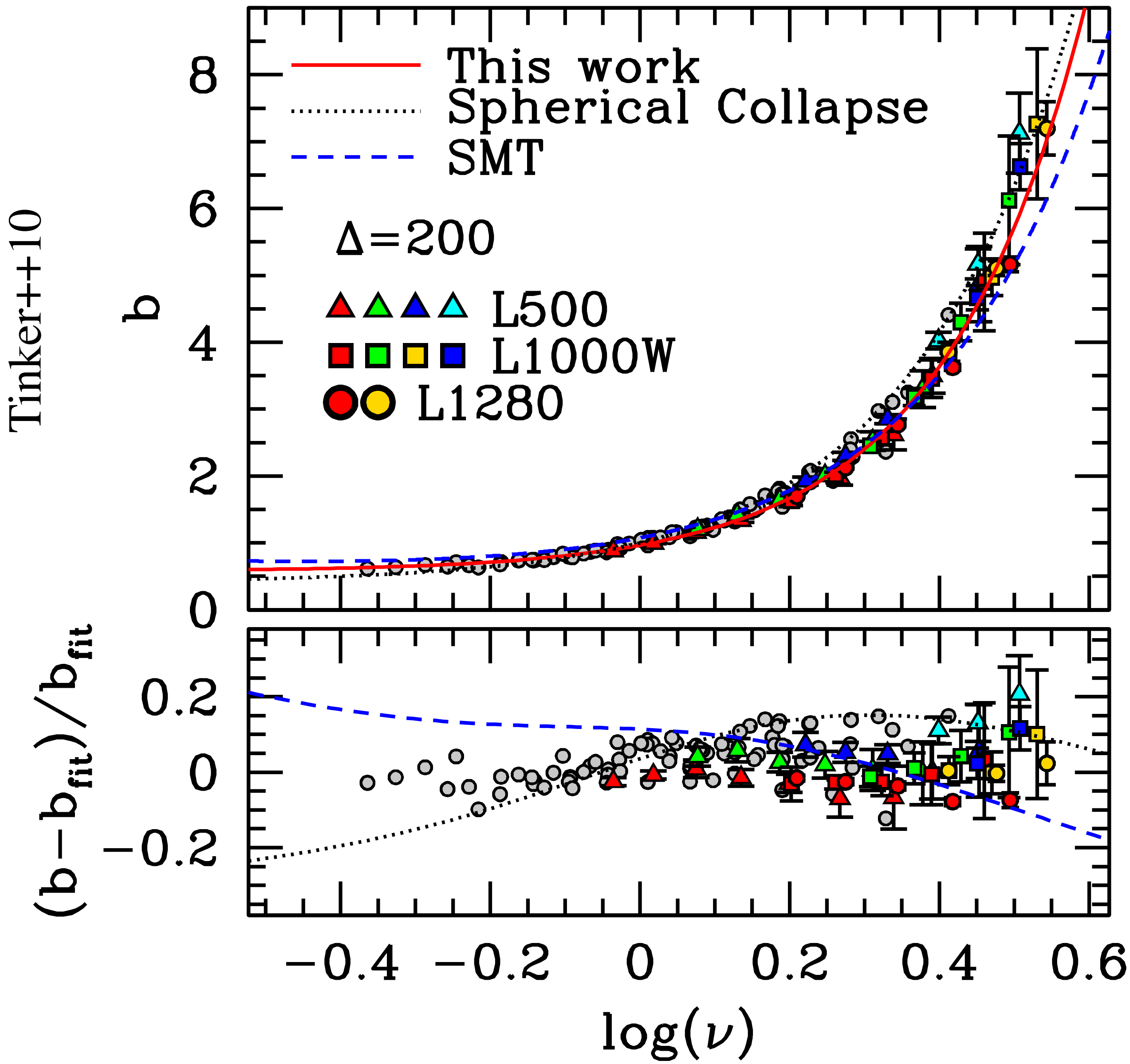


The clustering of rare halos thought to host quasars (here 10^{12} and $10^{12.5}$ M_{sun}/h) at z=3-4 is two orders of magnitude stronger than the clustering of the DM!

Halo bias

- This enhanced clustering is known as "bias".
- Bias depends on scale [$b(r)$], but at very large scales it becomes scale-independent [b].
  - Bias, b, depends primarily on halo mass or "rarity".
    - In simplest models $b=1+(\nu^2-1)/\delta_c$, where $\nu=\delta_c/\sigma(M)$.
    - For more accuracy, use N-body-calibrated fitting function.
    - Behavior at "extremes" can depart from fitting functions!
  - Numerical simulations now large enough to test for the dependence on halo formation history and other properties.
    - Dependencies on formation redshift, internal structure, and spin.
    - Gao++05; Wechsler++06; Harker++06; Bett++07; Wetzel++07; Jing++07; Gao&White07; Angulo++08

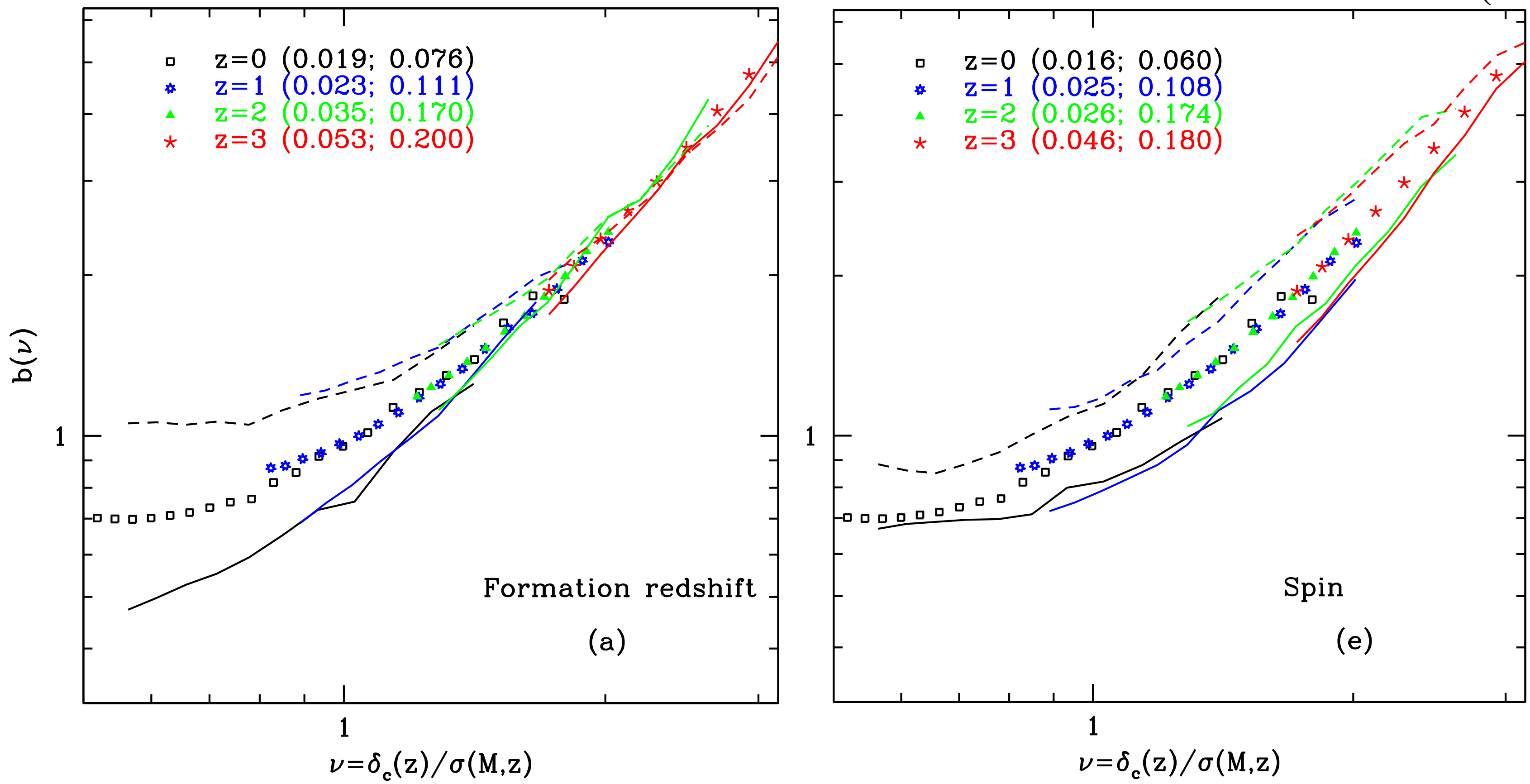
Halo bias in simulations



Halo bias increases with increasing halo mass at fixed redshift, or with increasing redshift at fixed mass.

Assembly bias

Gao & White (2007)



Solid (dashed) lines show halos in lower (upper) 20% of halos split on property labeled.

Assembly bias

- Assembly bias is quite difficult to explain in the "standard" excursion set formulation.
  - Mass function is fraction of random walks reaching an absorbing barrier by mass *M*.
  - Bias is dependence of mass function on large-scale density (early part of the walk).
  - Assembly bias very hard to explain in this picture.
    - Gao++05, Mo++05, Sandvik++07, Desjacques08, …
- Simulations did not initially shed light on explanation for assembly bias.
- Now understand that assembly bias is a simple consequence of non-linear collapse from Gaussian initial conditions.
  - Dalal++08.

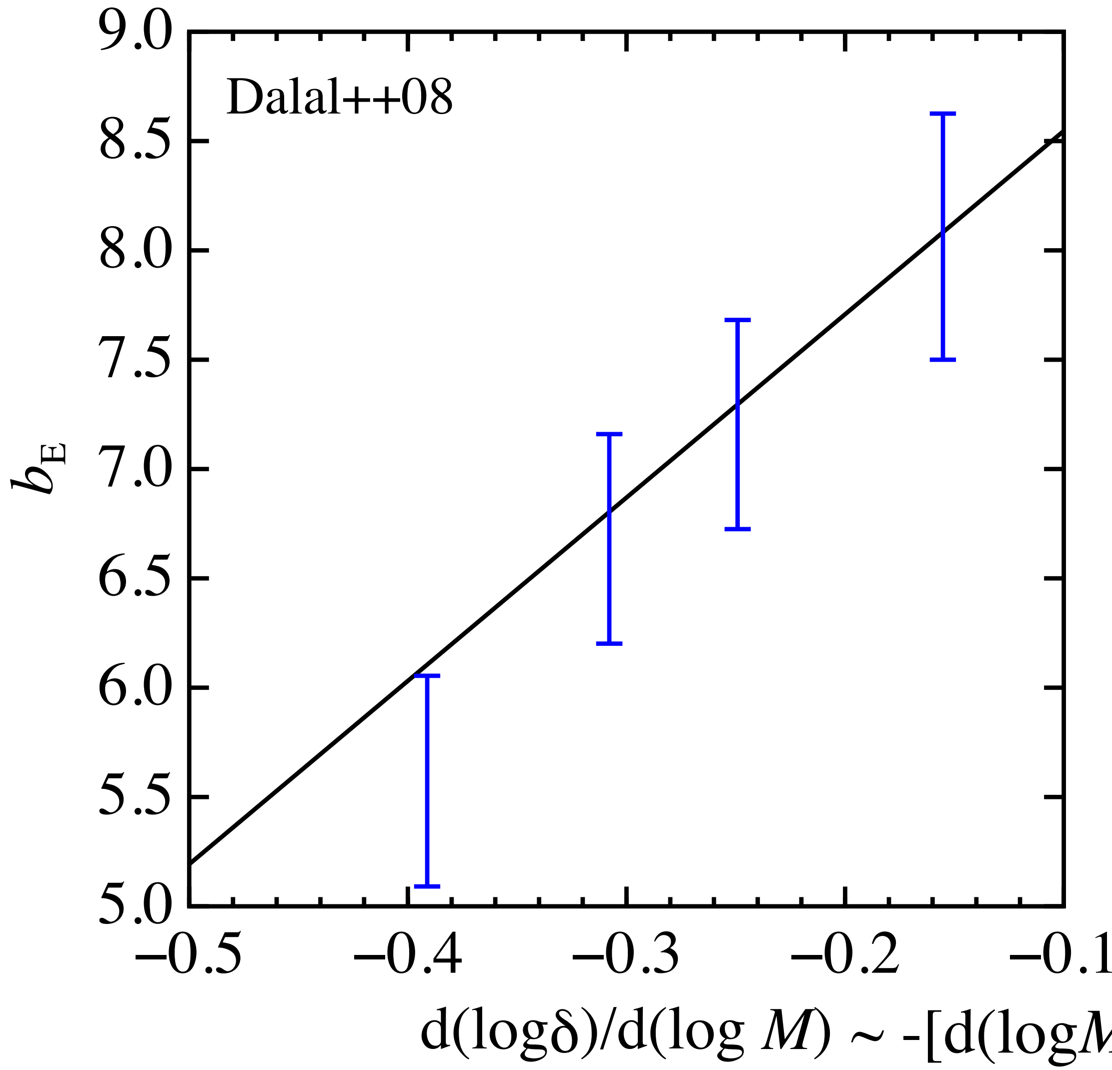
Assembly bias: high mass.

- Later forming, high mass halos are more clustered than typical halos of the same mass.
  - Also dependence on concentration.
- Massive halos collapse almost spherically from rare peaks in ICs.
  - Collapse reasonably explained by STHC.
- For Gaussian field, bias depends on curvature, $s=d\langle\delta\rangle/d\ln M$, of peak (as well as height).
  - Peak curvature is "environment": $\delta_b=\delta_{pk} + s\, d\ln M + \ldots$
  - Peaks with smaller $|s|$ have larger background densities.

$$b - 1 \approx \frac{1}{\sigma}\ \frac{\nu - \langle \nu x\rangle\, x}{1 - \langle \nu x\rangle^2} \quad , \quad \nu \equiv \frac{\delta}{\sigma_\delta}\ ;\ x \equiv \frac{s}{\sigma_s}$$

(Cross-correlation coefficient)

Bias: high mass

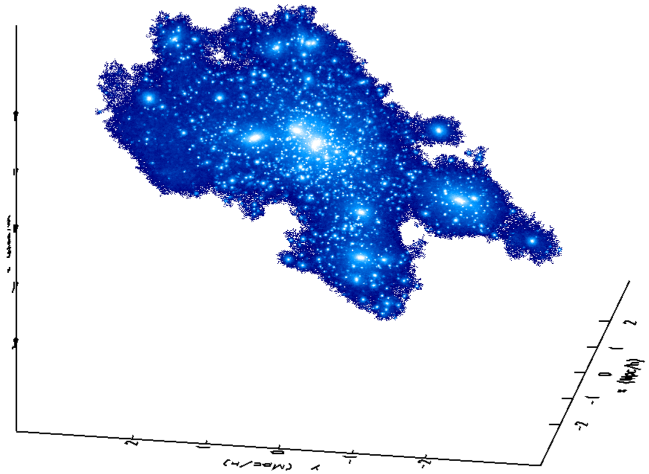


Dependence of halo bias on peak curvature from simulations (points) compared to the prediction from Gaussian peaks theory (line) for a power-law model. Assembly history related to run of δ with M – accretion rate related to peak curvature!

Assembly bias: low mass

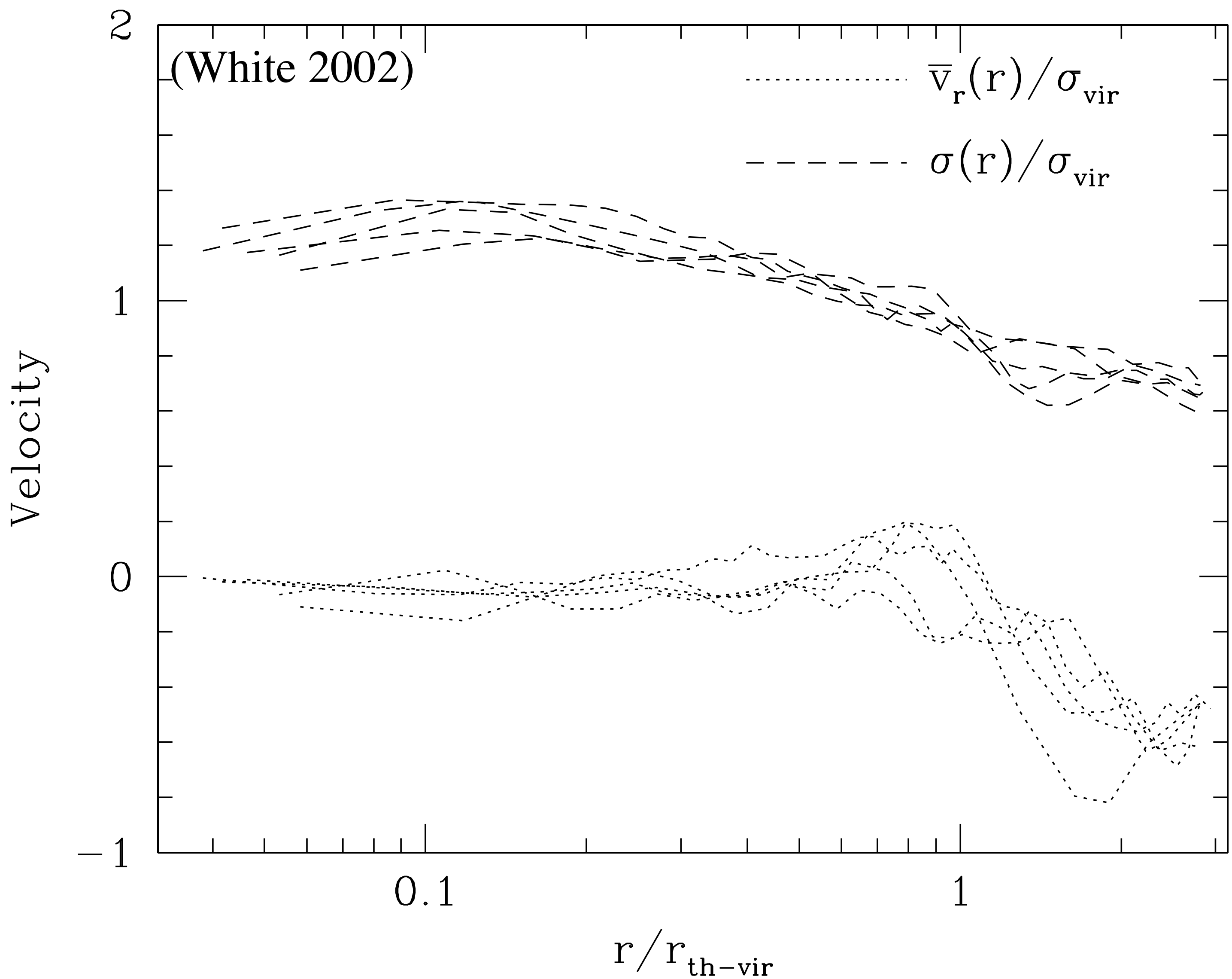
- Oldest, most concentrated, low mass halos are more than twice as clustered as the youngest halos of the same mass.
- Youngest ~80% of halos have
  - $b \sim 1-\delta_c^{-1} \sim 0.4$ (as expected).
- Oldest 20% of low mass halos act like test particles ($b \rightarrow 1$)
  - Most of these are associated with nearby, high-mass halos.
  - Early formers who's growth is stunted by "hot" environments of massive neighbors.

DM halos



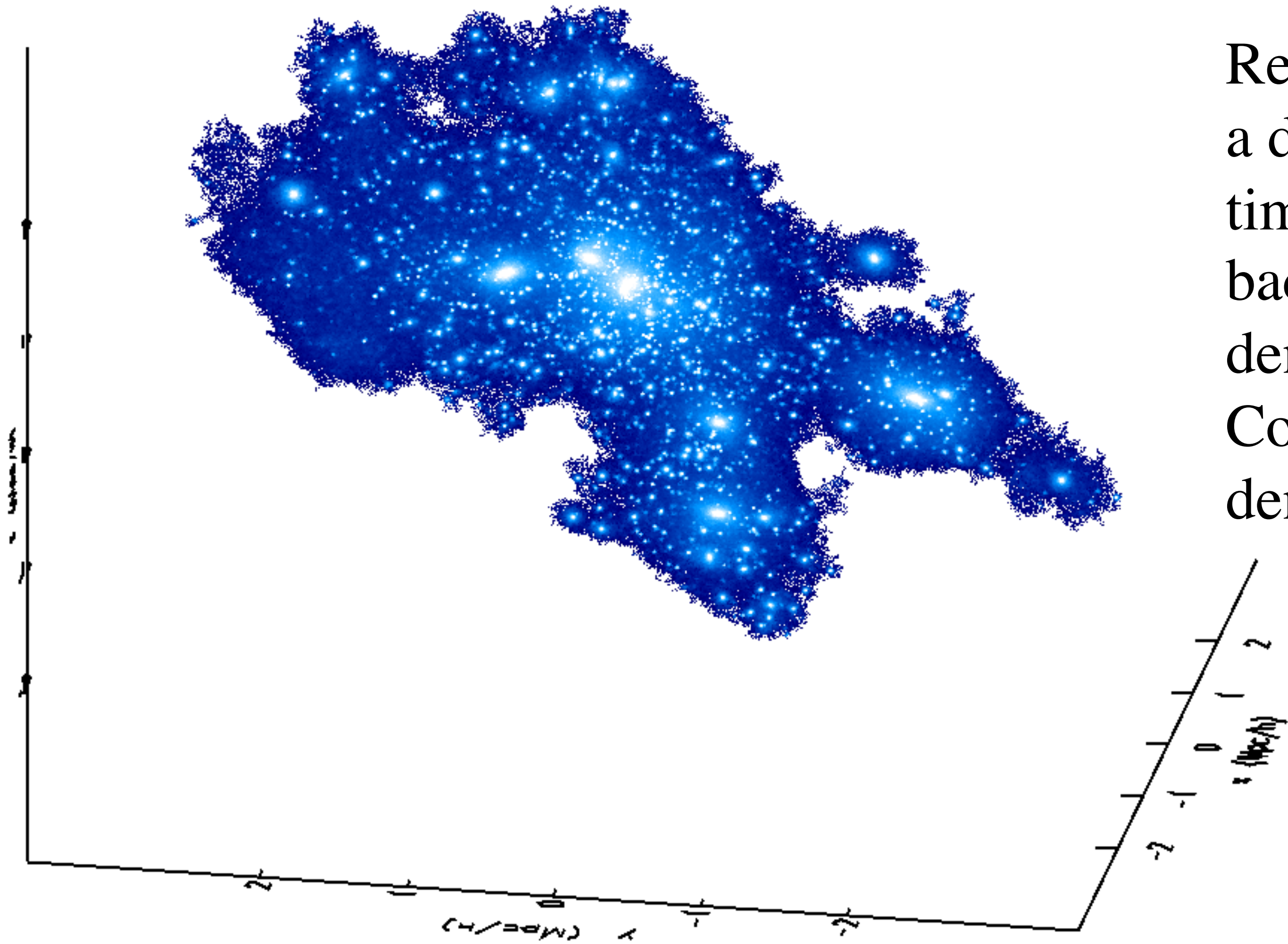
- Generally triaxial spheroids.
- More elongated at
  - Smaller radii.
  - Larger redshifts.
  - Higher mass.
- Approximately in virial equilibrium.
- Aligned with the filamentary, cosmic web which feeds halo growth.
- Average mass accretion exponential.
  - In EPS formalism $dM = -f(M)M\, d\delta_c$, with $f(M)$~constant.
- Spin parameter, λ, grows significantly in major mergers, slowly declines in accretion.

Dynamical state



Roughly isothermal, roughly virialized, self-bound objects.

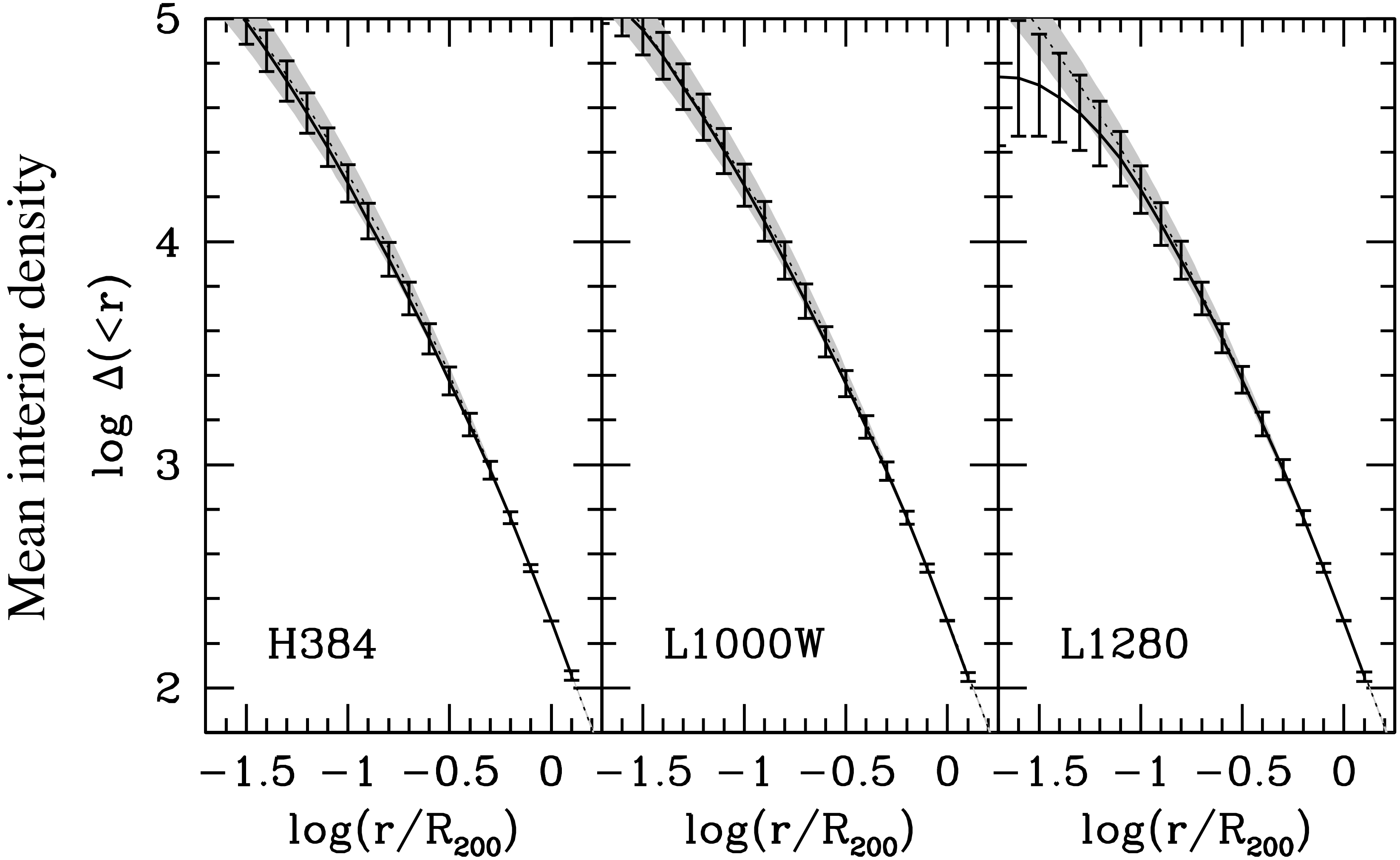
DM halos are aspherical and have significant substructure



Region above a density 10^2 times the background density. Color: log-density.

Spherical "NFW" profile

$$\rho(x = r/r_s) \propto x^{-1}(1+x)^{-2}$$



Tinker++08

A 1-parameter family

- Find $c=r_{vir}/r_s$ is a function of *M.*
  - More massive halos less concentrated.
  - *c*, like *M*, depends on definitions!
  - $c \sim M^{-0.15}$
  - Large, log-normal scatter in *c*.
- The inner, r^{-1}, part of the halo forms early and r_s stays ~constant.
  - Subsequent accretion kept away by angular momentum barrier.
  - Concentration, $c=r_{vir}/r_s \sim (1+z)^{-1}$.

Other forms

- A generalized NFW makes "-1" and "-2" variable.
- Einasto profile:

$$\rho \propto \exp\left\{-d_n\left[\left(\frac{r}{r_e}\right)^{1/n}-1\right]\right\} \quad , \quad n \approx 5-10$$

- Note no cusp!
- Important new insights in Lithwick & Dalal (2010).
  - Building on earlier work by Fillmore & Goldreich and Bertschinger.
- The NFW profile is "transitional".
  - r^{-3} slope comes from continued accretion of material. This stops in DE-domination.
  - Busha, Evrard & Adams (2007).
    - Exponential truncation of NFW profile at large radius.

Subhalos

- A generic prediction of hierarchical theories, such as CDM, is that the virialized regions of DM halos contain *subhalos.*
  - Self-gravitating, bound clumps of mass.
- Subhalos account for $O(10\%)$ of halo mass.
- Luminous galaxies form via the cooling and condensation of gas in subhalos.

Subhalos

- Density profiles of subhalos similar to that of halos, but they can be truncated.
- Subhalos track DM closely in terms of density and velocity.
  - Trends of central concentration and velocity bias with ratio of subhalo to host halo mass.
  - Depends on *how* subhalos are selected.
- Beyond a certain point, the number of subhalos above a given mass grows linearly with host halo mass.
  - Length of "plateau" set by dynamical friction and mean density of collapsed structures.
  - Subhalo mass function and halo mass function are "scaled" versions of each other.

$$\frac{dn_{\rm sat}}{dM_{\rm sat}} \sim \left(\frac{M_{\rm host}}{M_{\rm sat}}\right)^2 \qquad , \qquad M_{\rm sat} \ll M_{\rm host}$$

Thank you!

- I would like to thank
  - The participants.
  - The other lecturers.
  - The organizers.
  - The staff.
- for making this a pleasant, informative and productive meeting.

The End