The Zel'dovich Approximation

Large-scale structure goes ballistic

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Built on work done with Lile Wang (undergrad), Jordan Carlson (grad) and Beth Reid (postdoc).

Yakov Borisovich Zel'dovich

(8 March 1914 – 2 December 1987)

- The Zel'dovich approximation
- Computing the 2-pt function
  - Matter
  - Special cases
- Beyond real-space matter
- The ZSM/LSM
- Other statistics
- Conclusions

Zel'dovich approximation

- Following Jeans and Lifschitz, instability analysis in cosmology was initially formulated in an Eulerian way.
- Zel'dovich introduced a Lagrangian formulation.

Gravitational Instability: An Approximate Theory for Large Density Perturbations

YA. B. ZELDOVICH
Institute of Applied Mathematics, Moscow

Received September 19, 1969

An approximate solution is given for the problem of the growth of perturbations during the expansion of matter without pressure. The solution is qualitatively correct even when the perturbations are not small. Infinite density is first obtained on disc-like surfaces by unilateral compression.

The following layers are compressed first adiabatically and then by a shock wave. Physical conditions in the compressed matter are analysed.

Key words: Galaxies formation — Cosmology — Gravitational instability

Zel'dovich approximation

- Let us assume
  - $x = q + \Psi(q,t)$
  - and $\Psi(q,t)=A(t).\Psi(q)$
- Requiring that we reproduce linear theory, $\delta(x,t)\sim D(t)\delta(x)$, implies
  - $A(t)=D(t)$ and $\Psi\sim d\Phi\sim d(\ d^{-2}\ \delta\ )$
- Since

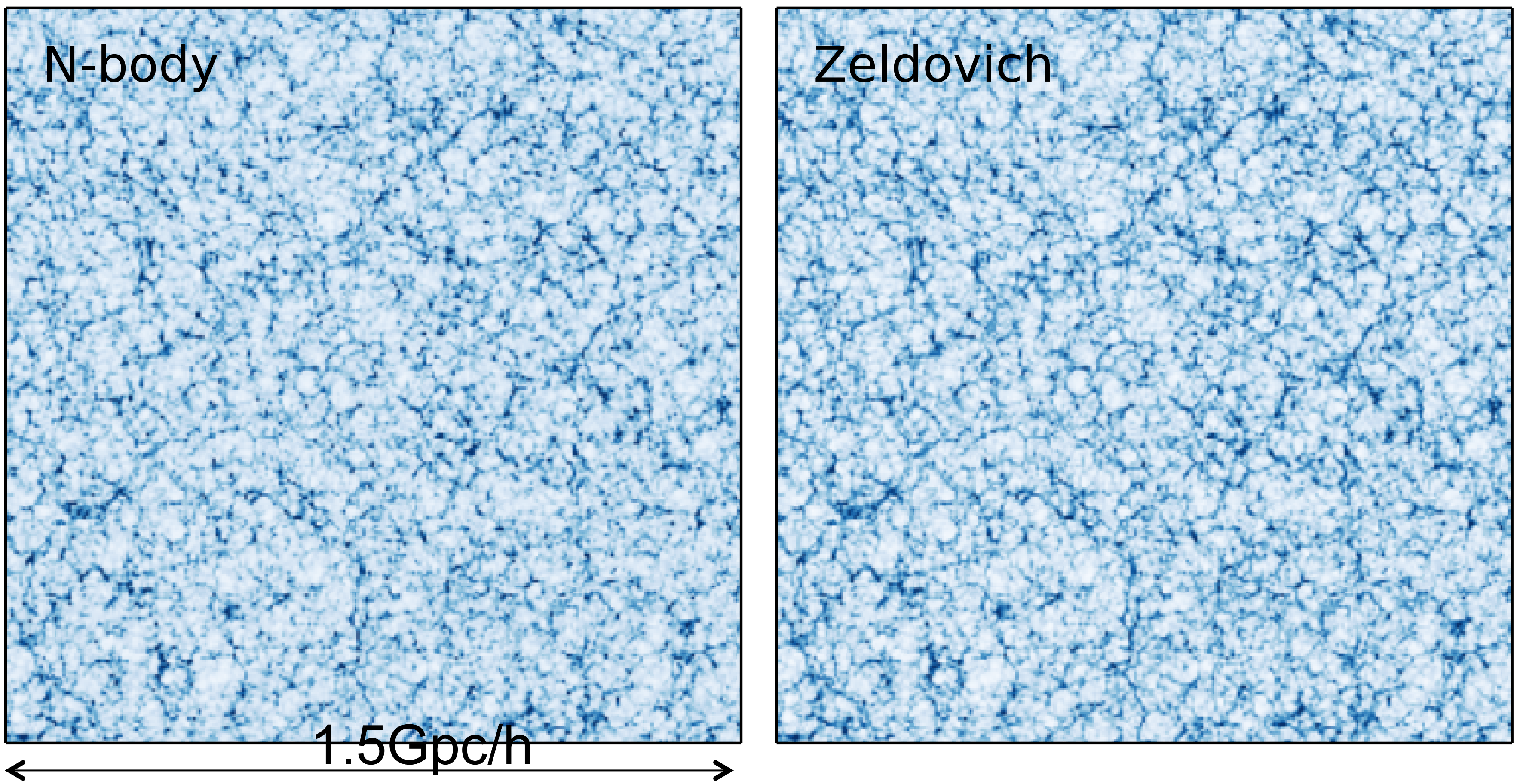
$$\rho(x,t)d^3x = \rho(q)d^3q \;\Rightarrow\; 1+\delta(x,t) = 1 - \nabla_q \cdot \Psi + \cdots$$

We assume Ψ retains this form always …

straight line (ballistic) motion!

Sheets, filaments & voids

The “cosmic web” of sheets, filaments and voids is the same in N-body simulations and Zel’dovich simulations …



Statistics of large-scale structure

- How well does the Zel'dovich approximation do quantitatively?
- Specifically, can we use it to compute the clustering of objects in the Universe?
  - Yes!
  - Can compute the correlation function of halos and galaxies, in real- and redshift-space with high accuracy to surprisingly small scales.
- Having a fully realized (though "wrong in detail") model of large-scale structure evolution enables "how does…" questions!

Like STHC …

Do the math …

Use a trick I learned from Matsubara's papers on Lagrangian perturbation theory …

$$
\begin{aligned}
1+\delta(x) &= \int d^3q\ \delta^{(D)}\left[\mathbf{x}-\mathbf{q}-\mathbf{\Psi}(\mathbf{q})\right] \\
&= \int d^3q \int \frac{d^3k}{(2\pi)^3}\ e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{q}-\mathbf{\Psi})}
\end{aligned}
$$

so

$$
\begin{aligned}
1+\xi(\mathbf{r}=\mathbf{x}_2-\mathbf{x}_1) &= \int d^3q_1 d^3q_2 \int \frac{d^3k_1}{(2\pi)^3}\frac{d^3k_2}{(2\pi)^3} \\
&\quad \times e^{i\mathbf{k}_1\cdot(\mathbf{x}_1-\mathbf{q}_1)} e^{i\mathbf{k}_2\cdot(\mathbf{x}_2-\mathbf{q}_2)} \\
&\quad \times \left\langle e^{-i\mathbf{k}_1\cdot\mathbf{\Psi}_1-i\mathbf{k}_2\cdot\mathbf{\Psi}_2}\right\rangle
\end{aligned}
$$

But Ψ is Gaussian …

- Cumulant theorem
  - For Gaussian x with $\langle x \rangle = 0$:
  - $\langle \exp[x] \rangle = \exp[-\frac{1}{2}\langle x^2 \rangle]$
- This allows us to rewrite our Gaussian integral as the exponential of $\langle \Psi\Psi \rangle$.

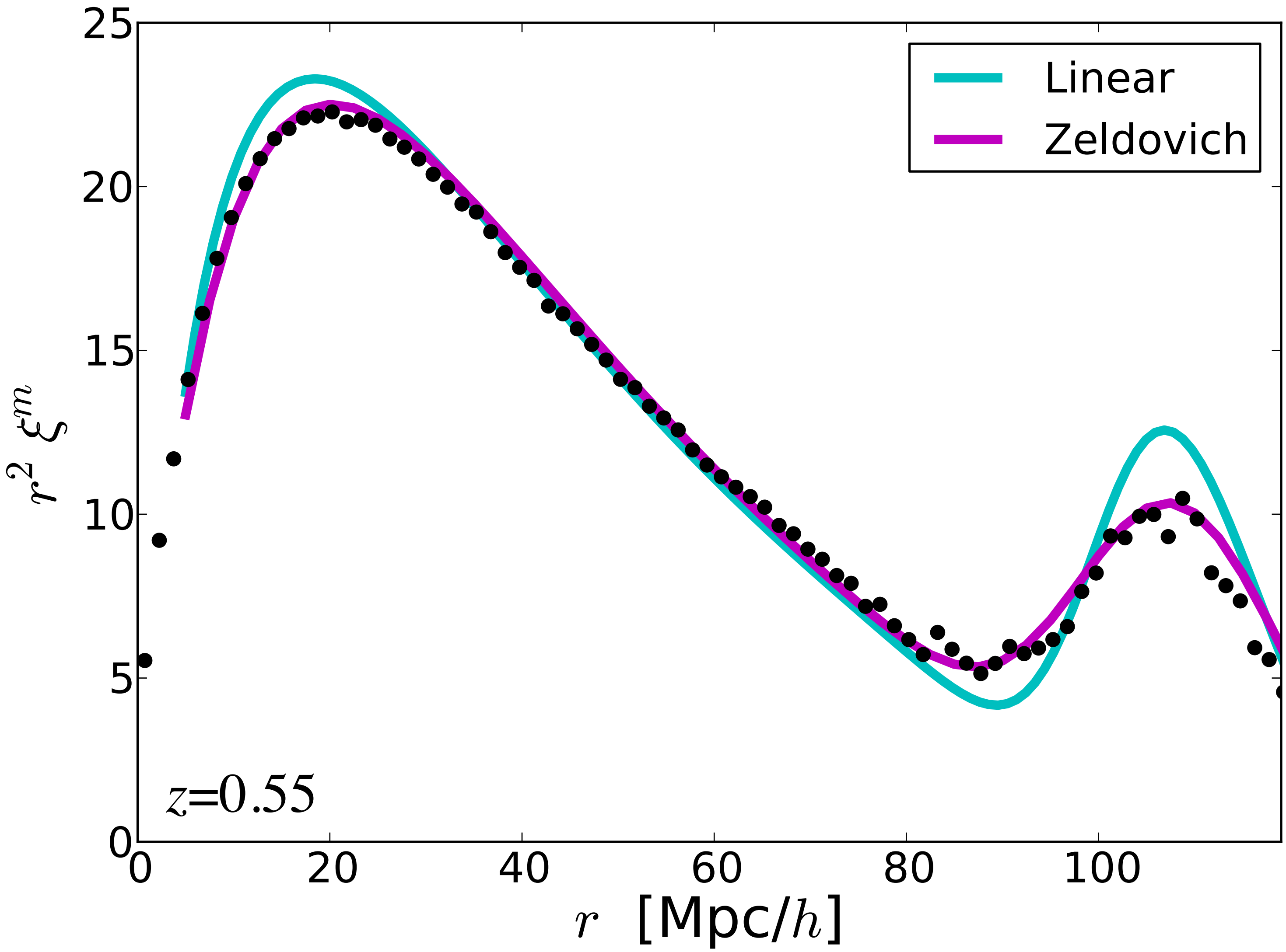
$$1 + \xi(\mathbf{r}) = \int \frac{d^3 q}{(2\pi)^{3/2} |A|^{1/2}} \times \exp\left[-\frac{1}{2}(\mathbf{r}-\mathbf{q})^T \mathbf{A}^{-1} (\mathbf{r}-\mathbf{q})\right]$$

$$\boxed{\mathbf{A}_{ij} = \langle [\Psi_i(\mathbf{q}_2) - \Psi_i(\mathbf{q}_1)] [\Psi_j(\mathbf{q}_2) - \Psi_j(\mathbf{q}_1)] \rangle}$$

The final result

- Can express A^{-1} and $|A|$ analytically.
- We have now reduced the calculation of the correlation function to the evaluation of a (simple, 3D) Gaussian integral.
  - The integrand contains simple, 1D integrals of the linear theory power spectrum.
- One of the integrals is trivial, so this is really a 2D integral – straightforward numerically.
  - A few seconds on a computer with the midpoint method: $\Sigma_i \, f(x_i) \, \Delta x$.
- One can also approximate the integral analytically …

(Dark) matter clustering



Massage?

- Can we massage this expression to get some intuition as to what's going on?
- To begin, we can split the pieces of A_{ij} that are *q*-independent from those that are *q*-dependent [$A_{ij}=B_{ij}+C_{ij}(q)$] and write

$$1+\xi = \int \frac{d^3q}{(2\pi)^{3/2}|B|^{1/2}} \times \ e^{-\frac{1}{2}(\mathbf{r}-\mathbf{q})^\top \mathbf{B}^{-1}(\mathbf{r}-\mathbf{q})}\left[1+\chi(\mathbf{q})\right]$$

- which is now *really* a convolution.
- We expect non-linearity to "smear" features.

Bharadwaj96; ESW07; Crocce&Scoccimarro08, Matsubara08; …

Splitting the bulk flows …

- Such a split appears natural, and it gives some insight.
- If we Taylor series expand the C pieces in powers of Ψ (keeping the zero-lag piece exponentiated) we find

$$P(k) \approx e^{-k^2\Sigma^2} P_L(k) + \cdots$$

- This type of form has been used extensively to model baryon acoustic oscillations, it is also the lowest order piece of the iPT or RPT schemes.
- But it does have drawbacks …

Extensions …

- If all we could compute was the 2-point function of the matter field in real space this would be *cool*, but of limited use.
- However … can also extend this formula to biased tracers such as halos or galaxies …
- … and to redshift space.
- This dramatically increases the range of problems where this method can teach us something valuable …
- … and it allows us a new window on some old problems.

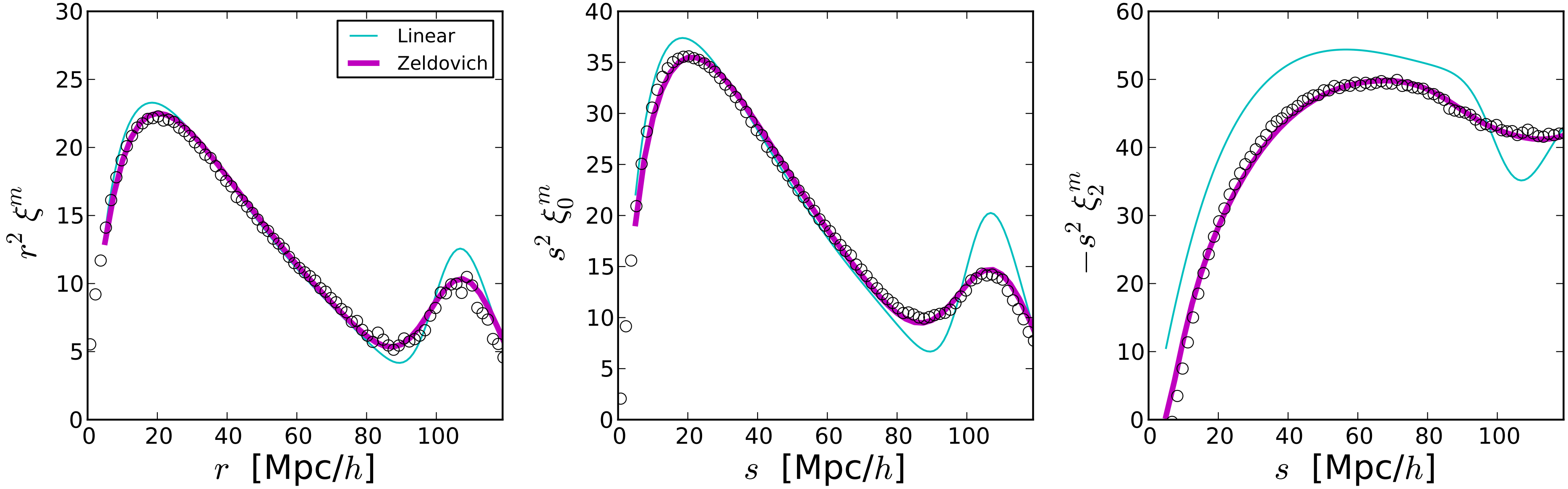
Beyond real-space mass

- One of the more impressive features of this approach is that it can gracefully handle both biased tracers and redshift space distortions.
- In redshift space, in the plane-parallel limit,

$$\mathbf{\Psi} \to \mathbf{\Psi} + \frac{\widehat{\mathbf{z}} \cdot \dot{\mathbf{\Psi}}}{H}\,\widehat{z} = R\,\mathbf{\Psi}$$

- In PT $\quad \mathbf{\Psi}^{(n)} \propto D^n \Rightarrow R^{(n)}_{ij} = \delta_{ij} + nf\,\widehat{z}_i\widehat{z}_j$
- i.e. multiply the line-of-sight component by 1+f.
- The lowest order terms return the usual Kaiser expression, the higher order terms give important modifications to this (since Kaiser's expression doesn't work very well!).

The dark matter …



Real space

Redshift space monopole

Redshift space quadrupole

Ok, but …

- Can we look at galaxies (and QSOs, and …) that live in halos?
- Not *ab initio*, since the Zel'dovich approximation won't form bound objects like halos.
- What about if we assume halos are simply biased tracers of the density field?
  – If the bias is local in Lagrangian space this is totally straightforward …
  – In more complex situations, we have directions we can explore …

Beyond real-space mass

For bias local in Lagrangian space:

$$\delta_{\rm obj}(\mathbf{x}) = \int d^3q \; F\left[\delta_L(\mathbf{q})\right] \delta_D(\mathbf{x} - \mathbf{q} - \mathbf{\Psi})$$

- which can be massaged with the same tricks as we used for the mass.
- If we assume halos/galaxies form at peaks* of the initial density field ("peaks bias") then explicit expressions for the integrals of F exist. Answers depend on bias parameters: b_n

$$b_n = \frac{1}{\nu f(\nu)} \; \frac{d^n}{d\delta^n}\left[\nu f(\nu)\right]$$

*…and assume the peak-background split.

Peaks bias

- Final result depends on averages of derivatives of F.
- The averages of F' and F'' over the density distribution take the place of "bias" terms
  - b_1 and b_2 in standard perturbation theory*.
- If we assume halos form at the peaks of the initial density field and use the peak-background split and assume the Press-Schechter mass function we can obtain:

$$b_1 = \frac{\nu^2 - 1}{\delta_c} \quad , \quad b_2 = \frac{\nu^4 - 3\nu^2}{\delta_c^2} \approx b_1^2$$

- with similar formulae in other cases (e.g. S-T).

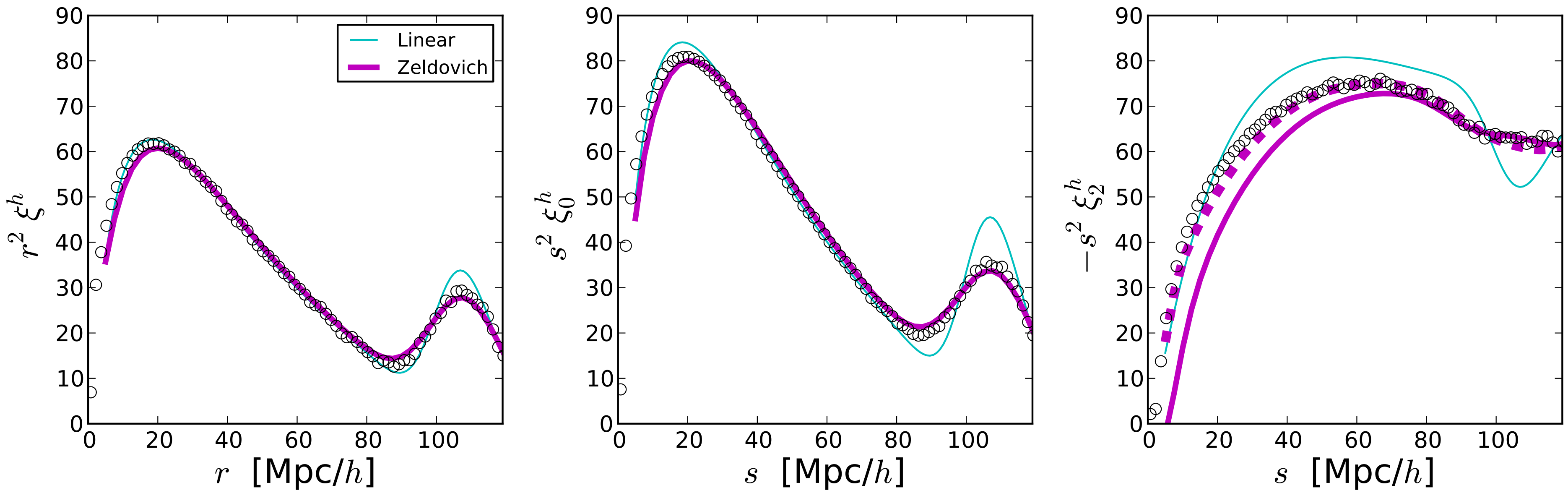
*but "renormalized".

Biased tracers …

$$
\begin{aligned}
1+\xi = \int & \frac{d^3q}{(2\pi)^{3/2}|A|^{1/2}} \\
& \times e^{-(1/2)(\mathbf{r}-\mathbf{q})^\top \mathbf{A}^{-1}(\mathbf{r}-\mathbf{q})} \\
& \times \Big[1 + b_1^2 \xi_L - 2b_1 U_i g_i + \frac{1}{2} b_2^2 \xi_L^2 \\
& - (b_2 + b_1^2) U_i U_j G_{ij} - 2 b_1 b_2 \xi_L U_i g_i + \cdots \Big]
\end{aligned}
$$

The “bias” we normally think of is $b=1+b_1$.

For (CMASS-like) halos



Real space

Redshift space monopole

Redshift space quadrupole

What’s up with ξ_2?

- For the mass, we got good agreement with real- and redshift-space ξ.
- For the halos, the monopole looks good but the quadrupole is “off”.
- Three possibilities come to mind:
  - The Zel’dovich approximation does less well around peaks or other specially chosen places*.
  - Local Lagrangian bias with the peak-background split is not a good description of halo bias in N-body simulations.
  - There are other terms, such as tidal shear, that are important to include.

*this is part of it …

Simplicity to the rescue …

- As the Zel'dovich approximation is so simple, it's possible to extend our bias calculation.
- Can consider an "effective field theory" approach where we put in all terms consistent with the symmetries (in a derivative expansion).
- This includes terms which have a "quadrupolar nature", like tidal shear:

$$\langle s^2 \rangle^2$$

$$s_{ij}(\mathbf{k}) = \left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij} \right) \ \delta(\mathbf{k})$$

$$\langle s_{ij}(\mathbf{q}_1)\delta(\mathbf{q}_2) \rangle^2$$

$$\langle s_{ij}(\mathbf{q}_1)s_{ij}(\mathbf{q}_2) \rangle^2$$

$$\langle s_{ij}(\mathbf{q}_1)k_m\Psi_m(\mathbf{q}_2) \rangle^2$$

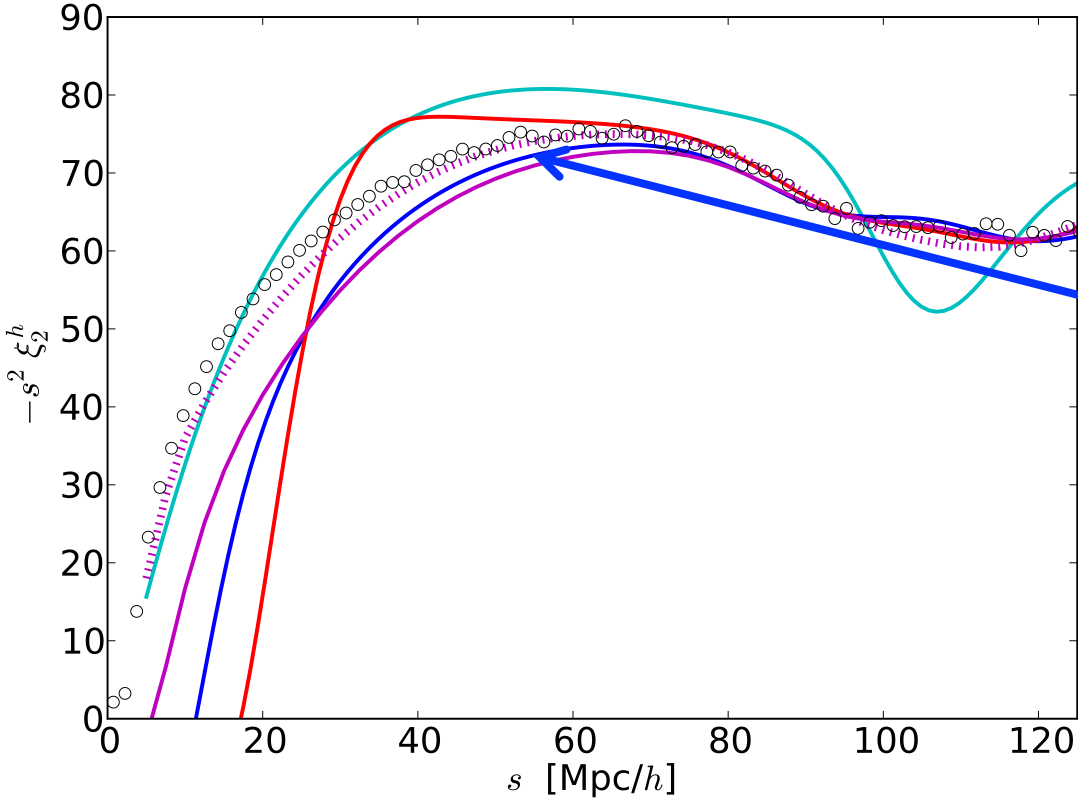
$$\langle s_{ij}(\mathbf{q}_1)\delta(\mathbf{q}_2) \rangle \langle s_{ij}(\mathbf{q}_1)k_m\Psi_m(\mathbf{q}_2) \rangle$$

Mixed results

- It is possible to get improvements in the results, but they aren't dramatic.
- In general these extra terms are quite small.
  - Note that the shear terms in Eulerian theory can be quite large, but those get contributions from the evolution.
- It would be interesting to try something very flexible but systematic ...

What about “better gravity”?

- Going to higher order in LPT improves the quadrupole, but only very slightly.
- At the cost of a lot of work (Carlson, Reid & White 2013)!



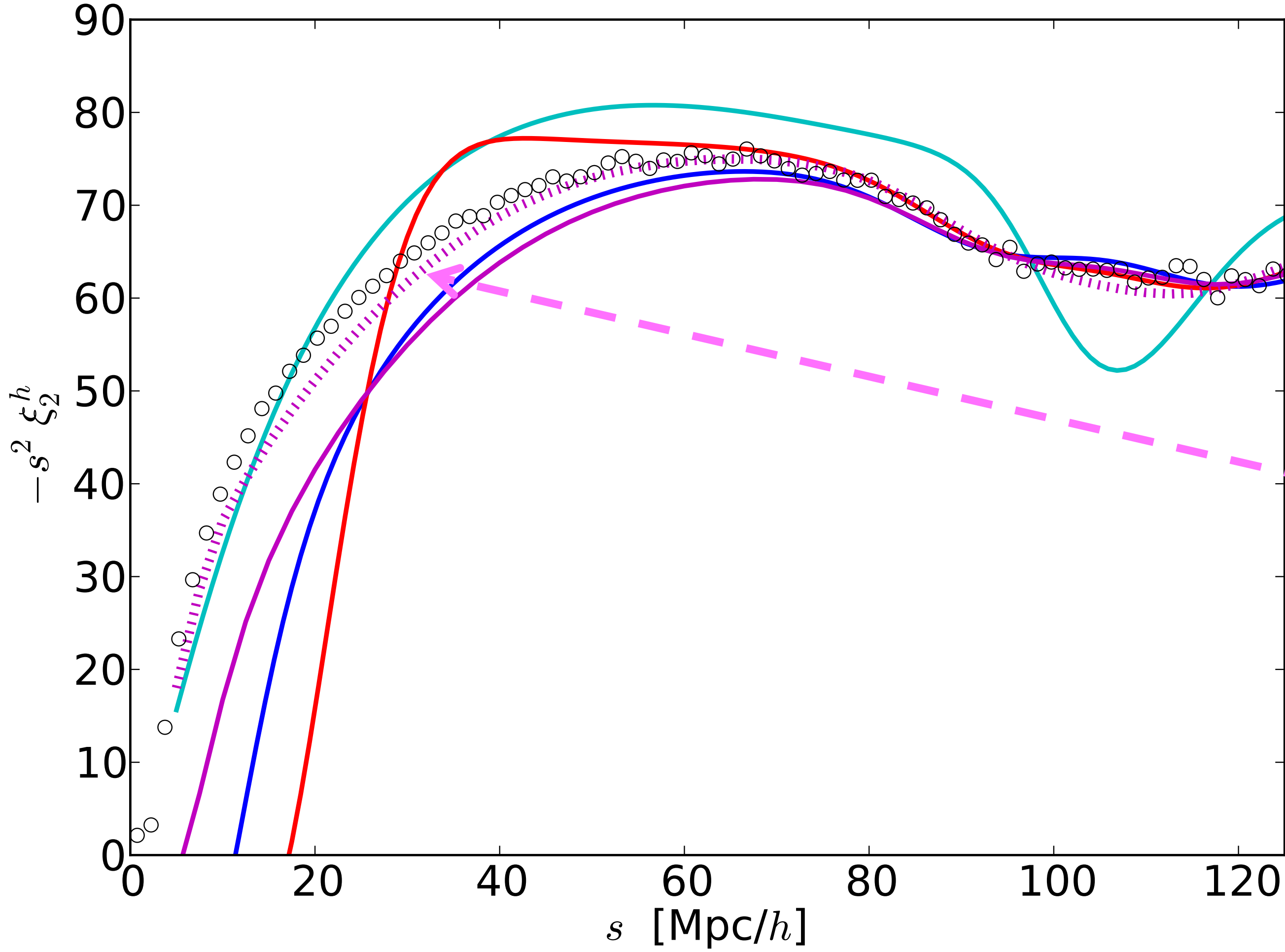
CLPT goes to second order in LPT consistently with the same type of resummation.

Zel'dovich streaming model

- Much of the failure of the Zel'dovich approximation lies in its prediction of pair-wise streaming velocities.
- N-body simulations show that for <u>halos</u>, the PDF of the pairwise velocities is close to Gaussian.
- What if we set the moments of the Gaussian using the Zel'dovich approximation, but the Gaussian form by fiat?
  - Zel'dovich streaming model.

see also Reid & White (2011); Wang, Reid & White (2013); …

Zel'dovich streaming model



ZSM does a better (though not perfect) job of fitting the quadrupole!

It turns out that going to higher order in LPT for v_{12} gives an excellent fit to N-body data … for halos and galaxies … (LSM)

Beyond the 2-point function?

- It is of course possible to go beyond the 2-point "object" auto-correlation function.
- Can go to higher orders, can look at cross-correlations, can look at non-linear mappings.
- Tassev has done the calculation for the 3-point function of the matter in real space.
- An efficient way of doing the calculation for the 4-point function of halos in redshift-space is still waiting to be found … though we've begun the exploration.
  - I can tell you several ways that don't work!

Conclusions

- The Zel’dovich approximation provides a numerically accurate, but surprisingly simple, approximation to large-scale structure statistics.
- Find good agreement for the real-space statistics, and the angle-averaged redshift-space correlations.
  - But not for the dependence on angle to the line-of-sight where it fails quite noticeably.

Can now test various models for how halo formation is related to initial conditions, the bias of peaks, the pair-wise velocity distribution of halos, cross-correlations, higher-order statistics, …

The End

Power-law models

- If the Universe had Ω_m=1 ($a \sim t^{2/3}$, $\delta \sim a$)
- and the initial power spectrum were a power law ($\Delta^2 \sim k^{3+n}$)
- then since gravity has no scale (a power-law potential) the resulting evolution would be self-similar.

$$\Delta^2(k) = \left(\frac{k}{k_\star}\right)^{3+n} \quad \Rightarrow \quad \xi(r) = \left(\frac{r}{r_\star}\right)^{-3-n} = B_n\,(k_\star r)^{-3-n}$$

$$\Delta^2(k,a) = F\left[k\,a^{2/(3+n)}\right] \quad \Rightarrow \quad k_\star \propto a^{-2/(3+n)}$$

Standard perturbation theory

- Of course all of the integrals in standard perturbation theory also become simple power-laws, e.g.

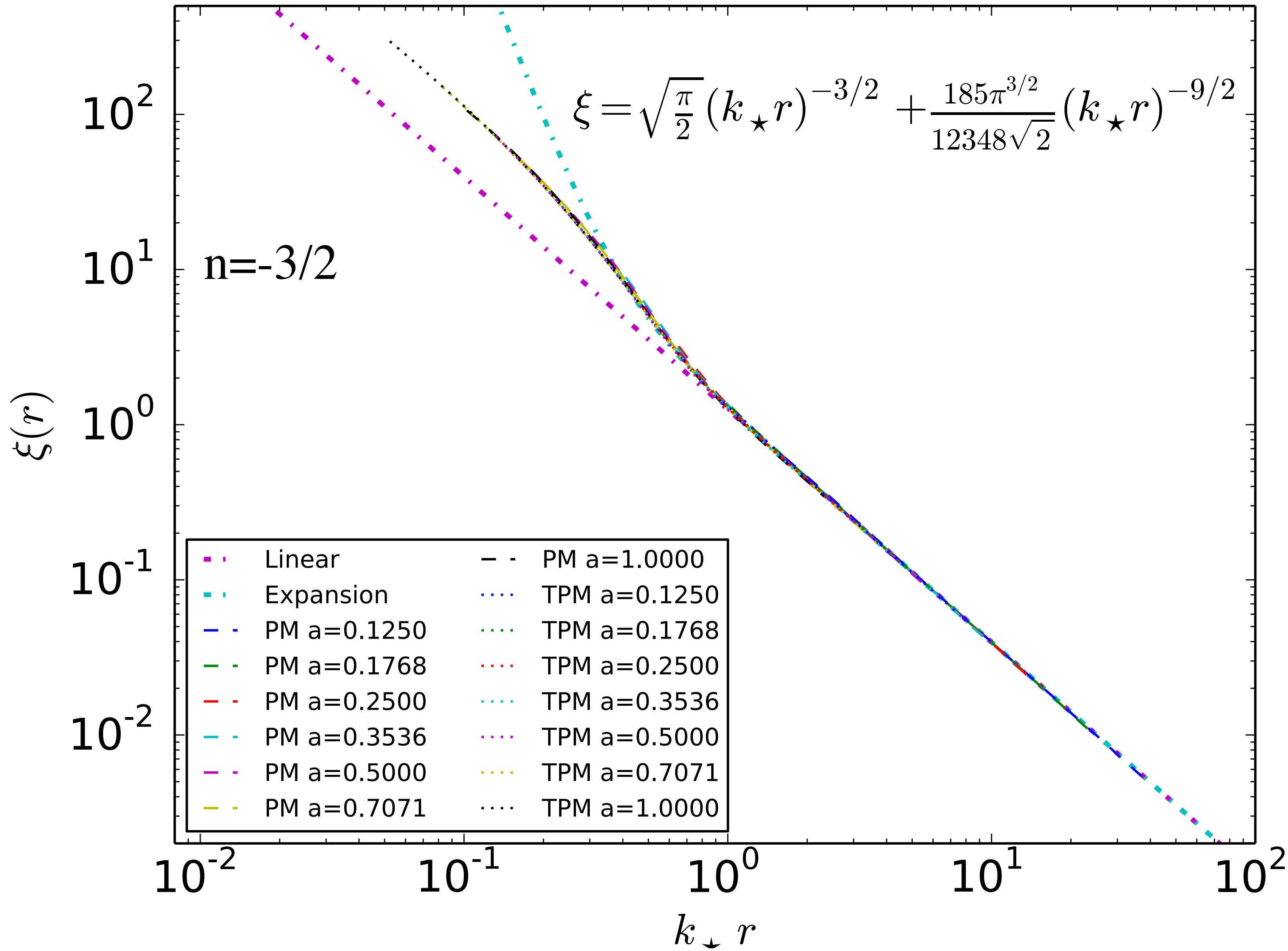
$$\Delta^2 = a^2\kappa\left[1 + \frac{110\,\pi^5}{98}a^2\kappa + \cdots\right] \qquad (n=-2)$$

- Unfortunately, this predicts a divergent correlation function (for any *r*).
- For "resummed" theories such as RPT or iPT the answer is zero …

What about Zeldovich?

- The dispersions are also power laws:
- So our Gaussian integral now becomes analytically tractable.
  - Integrate over $x=q-r$.
  - The “width” of the integration kernel is about the rms displacement.
  - Organize things as a power series in this displacement divided by r, i.e. $|x| \ll |r|$.

Power law


$\xi = \sqrt{\frac{\pi}{2}}(k_\star r)^{-3/2} + \frac{185\pi^{3/2}}{12348\sqrt{2}}(k_\star r)^{-9/2}$
n=-3/2
$\xi(r)$
$k_\star r$
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Linear
Expansion
PM a=0.1250
PM a=0.1768
PM a=0.2500
PM a=0.3536
PM a=0.5000
PM a=0.7071
PM a=1.0000
TPM a=0.1250
TPM a=0.1768
TPM a=0.2500
TPM a=0.3536
TPM a=0.5000
TPM a=0.7071
TPM a=1.0000

What's different?

- In the Zeldovich approximation, what matters is the rms displacement between two points initially separated by q.
- In RPT, iPT or other schemes what matters is the rms displacement at a <u>single</u> point (which is q independent: Σ).
- For many power-law models the former is finite while the latter is not!
- While the power-law models are extreme in this sense, they point to a very important point about "splitting" bulk flows.

see also Tassev & Zaldarriaga