Low order statistics of shifted fields

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Shifted fields

- There are two situations in cosmology where the statistics of "shifted" fields are of interest:
  - Redshift-space distortions.
  - Reconstruction (for BAO).
- Many aspects of these problems are common, allowing a unified treatment.
- While many of the results below will be quite general, where necessary I will illustrate it with LPT+EFT.

Background: LPT

- To fix notation, recall a fluid element (or DM particle) initially at $\mathbf{q}$ at $t_{\rm init}$ moves to $\mathbf{x}(\mathbf{q},t) = \mathbf{q} + \boldsymbol{\Psi}(\mathbf{q},t)$ with $\ddot{\boldsymbol{\Psi}} + \mathcal{H}\dot{\boldsymbol{\Psi}} = -\nabla\Phi(\mathbf{q}+\boldsymbol{\Psi})$.
- We necessary we shall solve for $\boldsymbol{\Psi}$ perturbatively, including contributions from unmodeled small-scale physics using counterterms.
- The $1^{\rm st}$ order solution is the Zeldovich approximation.
- Given $\boldsymbol{\Psi}$, the Eulerian density is

$$1+\delta(\mathbf{x}) = \int d^3q\ \delta^{(D)}\left[\mathbf{x}-\mathbf{q}-\boldsymbol{\Psi}(\mathbf{q})\right]$$

or

$$\delta(\mathbf{k}) = \int d^3q\ e^{i\mathbf{k}\cdot\mathbf{q}}\left(e^{i\mathbf{k}\cdot\boldsymbol{\Psi}(\mathbf{q})}-1\right)$$

Including Lagrangian bias is straightforward, but due to time constraints I will focus on the matter to illustrate the principles.

Background: shifted fields

- For RSD, shift arises due to physics: $\mathbf{s} = \mathbf{x} + \hat{n}\,(\mathbf{v}\cdot\hat{n}/\mathcal{H})$.
- For reconstruction shift is applied by analyst, and is some functional of the observed (biased, non-linear, redshift-space) density.
  - For illustration I will use the 'traditional' algorithm involving the Zeldovich approximation.
  - The generalization to other algorithms is under study.
- Aside from the source of the shift, the modeling of the shifted fields presents many common elements:
- Introduce a 'shift' field χ which displaces objects from $\mathbf{q}+\mathbf{\Psi}$ to $\mathbf{q}+\mathbf{\Psi}+\chi$.
- To keep notation easy, let $\chi = \chi(\mathbf{x})$.

Shifting I

- The density field is

$$(2\pi)^3\delta^{(D)}(\mathbf{k}) + \delta_\chi(\mathbf{k}) = \int d^3q\ \exp\left[i\mathbf{k}\cdot\{\mathbf{q}+\mathbf{\Psi}+\chi\}\right]$$
$$= \int d^3x\ [1+\delta(\mathbf{x})]\exp\left[i\mathbf{k}\cdot\{\mathbf{x}+\chi\}\right]$$

- This is the FT of $1+\delta(\mathbf{x})$ times a phase, $\exp[i\mathbf{k}\cdot\chi]$.
  - Shift = phase
- Note if you go through Lagrangian coordinates you don't need to assume a single-stream, or 1-to-1 approximation.

Shifting II

- The 2-point function of $\delta_\chi(\mathbf{k})$ will have $1+\delta(\mathbf{x}_1)$ times $1+\delta(\mathbf{x}_2)$ and an exponential of $\Delta\chi = \chi(\mathbf{x}_2) - \chi(\mathbf{x}_1)$.
- Define the moment generating function of the shifted fields

$$1 + \mathcal{M}(\mathcal{J}, \mathbf{r}) = \left\langle [1+\delta(\mathbf{x}_1)]\,[1+\delta(\mathbf{x}_2)]\, e^{i\mathcal{J}\cdot\Delta\chi} \right\rangle$$

  where $\mathbf{r} = \mathbf{x}_2 - \mathbf{x}_1$.
- Note $\mathcal{M}$ and Ξ depend only upon $\mathbf{r}$!
- $\mathcal{J}-$derivatives of $\mathcal{M}$ (at $\mathcal{J}=0$) give density-weighted moments of $\Delta\chi$:

$$\Xi_{i_1,\ldots,i_n}(\vec{r}) = \left\langle \big(1+\delta(\vec{x}_1)\big)\big(1+\delta(\vec{x}_2)\big)\Delta\chi_{i_1}\ldots\Delta\chi_{i_n} \right\rangle$$

Shifting III

- If we FT $\mathcal{M}$ on $\mathbf{r}$ we have (up to a $\delta^{(D)}$):

$$\widetilde{\mathcal{M}}(\mathcal{J},\mathbf{k}) = \int d^3r\ e^{i\mathbf{k}\cdot\mathbf{r}} \left\langle [1+\delta(\mathbf{x}_1)]\,[1+\delta(\mathbf{x}_2)]\, e^{i\mathcal{J}\cdot\Delta\chi} \right\rangle$$

- Immediately we see that the power spectrum is

$$P(\mathbf{k}) = \widetilde{\mathcal{M}}(\mathcal{J}=\mathbf{k},\mathbf{k}) = \int d^3r\ e^{i\mathbf{k}\cdot\mathbf{r}}\mathcal{M}(\mathbf{J}=\mathbf{k},\mathbf{r})$$

- This relation holds beyond perturbation theory!
- To get ξ we need to do another FT.

Note we need an extra FT to get ξ regardless of whether we start from $\mathcal{M}$ or $\widetilde{\mathcal{M}}$.

Four methods

There are 4 routes forward, which have been pursued extensively in the literature but can be viewed within this same language:

1. Direct Lagrangian approach
   - Transform $\mathcal{M}$ to Lagrangian coordinates and use cumulant theorem.
2. Moment expansion approach.
   - Expand $\mathcal{M}$ and evaluate moments using SPT or use DF approach.
3. Streaming model(**s**).
   - Transform $\mathcal{M}$ to cumulant generating function, $\mathcal{Z}$. There are **two** inequivalent streaming models.
4. Smoothing kernel (Scoccimarro).
   - Multiply out $(1+\delta)(1+\delta)e^{\cdots}$ and apply cumulant theorem to each contribution to $\mathcal{M}$.
   - Forms basis of "TNS" method, which further expands the exponential and approximates the kernel as a Gaussian.

Four methods

- Labels and classes are primarily historical.
- All methods would be identical if carried to the same order with the same approximations.
- At finite order, they resum various things and some may be better for some purposes/models than others.
- It is worth stressing that the streaming model is a truncation of the cumulant expansion, not a *phenomenological* model.
- All 'models' are valid beyond shell crossing – it is when we do 'traditional' PT expansions that we lose that validity (EFT c.t.).
- This view provides a nice link between Eulerian moments and cumulants and Lagrangian dynamics, which can be useful.

Direct Lagrangian approach I: RSD

Transform $\mathcal{M}$ to Lagrangian coordinates and use cumulant theorem.
For example, for RSD:

$$P^s(\mathbf{k}) = \int d^3q \; e^{i\mathbf{k}\cdot\mathbf{q}} \left\langle [1+\delta(\mathbf{q}_1)]\,[1+\delta(\mathbf{q}_2)] \right.$$
$$\left. \times \; \exp\left[i\mathbf{k}\cdot(\mathbf{\Delta\Psi}+\mathbf{\Delta v})\right] \right\rangle$$

This leads to the formulae for iPT or CLPT and their extensions.

For illustration consider the Zeldovich approximation – 1^{st} order LPT – for the matter, though nothing in the formalism requires these simplifications.

Direct Lagrangian approach II: RSD

For the matter everything depends upon the Lagrangian 2-point function

$$A_{\ell m}(\mathbf{q}) = \langle \Delta_\ell \Delta_m \rangle \quad \text{with} \quad \Delta_i = \Psi_i(\vec{q}) - \Psi_i(\vec{0})$$

which for Zeldovich can be expressed in terms of integrals over the linear theory power spectrum. Writing

$$\begin{aligned} A_{ij}(\mathbf{q}) &= X(q)\delta_{ij} + Y(q)\hat{q}_i\hat{q}_j \\ &= \tfrac{2}{3}\delta^K_{ij}\left[\mathcal{J}_0(0) - \mathcal{J}_0(q)\right] + 2\left(\hat{q}_i\hat{q}_j - \tfrac{1}{3}\delta^K_{ij}\right)\mathcal{J}_2(q) \end{aligned}$$

we have

$$\mathcal{J}_0(q) = \int_0^\infty \frac{dk}{2\pi^2}\, P_L(k) j_0(kq) \quad \text{and} \quad \mathcal{J}_2(q) = \int_0^\infty \frac{dk}{2\pi^2}\, P_L(k) j_2(kq)$$

Direct Lagrangian approach III: RSD

The Zeldovich approximation power spectrum is then

$$P^s(\mathbf{k}) = \int d^3q \ \exp\left[-\frac{1}{2}k_i k_j A_{ij}(\mathbf{q})\right]$$

or

$$P^s(k,\nu) = 4\pi \sum_{n=0}^{\infty} \int q^2\, dq\, \mathcal{K}_n^s(k,\nu,q) e^{-(1/2)k^2[X+Y]} \left(\frac{kY}{q}\right)^n j_n(kq)$$

where $\mathcal{K}$ has an unenlightening (but very cool!) expression in terms of hypergeometric functions. First calculation of $P_{\rm Zel}^{\rm red}$ we know of!

Zeldovich RSD power spectrum (Vlah special)

$$\mathcal{K}^s_n(k,q,\nu) = \left(1+f\nu^2\right)^{2n} e^{-(1/2)f\nu^2k^2\left[(2+f)X(q)+(2+f\nu^2)Y(q)\right]} \times K^s_n\left(\nu, -\tfrac{1}{2}a_1k^2Y\right)$$

with

$$K^s_n(\nu,x) = \sum_{\ell=0}^{\infty}(-1)^\ell F_\ell(\nu,x)U(-\ell, n-\ell+1, -x),$$

$$F_\ell(\nu,x) = \sum_{m=0}^{\ell}\frac{(-1)^m 4^{\ell-m}\Gamma(m+\frac{1}{2})}{\pi^{1/2}\Gamma(m+1)\Gamma(1+2m-\ell)\Gamma(2\ell-2m+1)}\left(\frac{a_0}{a_1}\right)^m \times M\left(\ell-2m;\ell-m+\tfrac{1}{2};x\right) M\left(m+\tfrac{1}{2};m+1;\frac{a_0}{a_1}x\right),$$

with $a_0(\nu)=f^2\nu^2(1-\nu^2)$, $a_1(\nu)=\left(1+f\nu^2\right)^2$ and $M(a,b,z)$ and $U(a,b,z)$ the confluent hypergeometric function of the $1^{\rm st}$ (Kummer's) and $2^{\rm nd}$ (Tricomi's) kind respectively.

Cumulant approach: **two** streaming models

Expand $\mathcal{Z} = \ln[1+\mathcal{M}]$ in powers of $\mathcal{J}$, in configuration space:

$$\mathcal{Z}_\chi(\vec{J},\vec{r}) = \sum_{n=0}^{\infty} \frac{i^n}{n!} J_{i_1} \ldots J_{i_n} \mathcal{C}^{(n)}_{i_1 \ldots i_n}(\vec{r}),$$

The first few cumulants are

$$\begin{aligned} \mathcal{C}^{(0)}(\vec{r}) &= \ln\left[1+\xi(\vec{r})\right], \\ \mathcal{C}^{(1)}_i(\vec{r}) &= \frac{\Xi_i(\vec{r})}{1+\xi(\vec{r})} = \frac{\langle(1+\delta)(1+\delta)\Delta\chi_i\rangle}{1+\xi} = v_{12} \quad \text{, etc.} \end{aligned}$$

Plugging in to get $\mathcal{M}$ and doing a FT one obtains

$$P^s(\mathbf{k}) = \int d^3r \; e^{i\mathbf{k}\cdot\mathbf{r}} \left[1+\xi\right] \exp\left[\sum_{n=1}^{\infty} \frac{i^n}{n!} k_{i_1} \cdots k_{i_n} \mathcal{C}_{i_1,\cdots,i_n}(\mathbf{r})\right]$$

Keeping up to 2^{nd} order and doing Gaussian integral gives GSM.

Cumulant approach: **two** streaming models

The non-linearity of the cumulant expansion implies there are **two** streaming models, making two sets of predictions for ξ and P.

Expand $\mathcal{Z} = \ln[1 + \mathcal{M}]$ in powers of $\mathcal{J}$, in Fourier space:

$$\begin{aligned} \widetilde{\mathcal{C}}^{(0)}(\vec{k}) &= \ln\left[P(\vec{k})\right], \\ \widetilde{\mathcal{C}}_i^{(1)}(\vec{k}) &= \widetilde{\Xi}_i(\vec{k})/P(\vec{k}), \quad \text{etc.} \end{aligned}$$

Now the relationship between P in real and redshift space is algebraic:

$$P^s(\mathbf{k}) = P\widetilde{\mathcal{K}} = P \exp\left[\sum_{n=1}^{\infty} \frac{i^n}{n!} k_{i_1} \cdots k_{i_n} \widetilde{\mathcal{C}}_{i_1,\cdots,i_n}(\mathbf{k})\right]$$

Fourier streaming model

- The cumulants are constructed from the moments, and at any order the Fourier and configuration space moments contain the same information (they are FT pairs).
- The Fourier space SM can still be more convenient: one less step.
  - Connection between real and redshift space is algebraic!
- If you expand the exponential in $\mathcal{K}$ you get the moment expansion – so FoG are spread over all moments.
- **But** for SM these terms are resummed in a convenient way/compact form.
  - We know FoG are 'problematic' non-linear terms. Such big terms can be easily kept in the exponential in a clear way.
- Nice connection to 'old' dispersion models where $\mathcal{K}$ was taken to be $\exp[-k_\parallel^2\sigma^2]$ or $(1+k_\parallel^2\sigma^2)^{-1}$.

Streaming model $\leftrightarrow$ dispersion model

- Nice connection to 'old' dispersion models where $\mathcal{K}$ was taken to be $\exp[-k_\parallel^2\sigma^2]$ or $(1+k_\parallel^2\sigma^2)^{-1}$.
- To quadratic order

$$P^s(\mathbf{k}) = P(k)\exp\left[ik_\parallel \underbrace{\widetilde{\mathcal{C}}^{(1)}(\mathbf{k})}_{v_{12}} - \frac{1}{2}k_\parallel^2 \underbrace{\widetilde{\mathcal{C}}^{(2)}(\mathbf{k})}_{\sigma_{12}^2}\right]$$

- In PT the $\Xi_{ij} = \langle(1+\delta)(1+\delta)u_i u_j\rangle$ expression contains a term going as $P_L \int P_L$, which is UV-sensitive.
- This gives a contribution to $\widetilde{\mathcal{C}}^{(2)}$ that looks like a constant.
- i.e. a piece of $\mathcal{K}$ is $\exp[-k_\parallel^2\,\mathrm{const}^2]$.
- But can also compute corrections self-consistently.

Reconstruction

- Density field reconstruction is used to improve the performance of BAO by sharpening the acoustic peak in ξ.
- In 'standard' reconstruction, the shift field is computed from the observed, redshift-space density field under the Zeldovich approximation:

$$\chi_{\mathbf{k}} = -\frac{i\mathbf{k}}{k^2}\left(b + f\nu^2\right)^{-1}\mathcal{S}(k)\ \delta(\mathbf{k})$$

  where $\mathcal{S}$ is a smoothing kernel (e.g. a Gaussian).
- For modeling the reconstruced P or ξ the direct LPT approach, (to lowest order in $\mathbf{\Psi}$) has been computed, and Eulerian versions exist.
- Can now systematically develop these schemes to arbitrary order [expand in $\chi(\mathbf{x}) - \chi(\mathbf{q})$].

Can 'recycle' all of the development of these RSD models to models of reconstruction.

Conclusions

- Statistics of redshift-space or reconstructed fields can be treated in a consistent manner
  - Steal methods from one field for another.
- Four approaches, depending upon how one computes $\mathcal{M}$.
  - I focused on two here, because of time, but the others are also very useful and powerful.
- An explicit calculation of the redshift-space power spectrum in the Zeldovich approximation.
- A new variant of the streaming model in which the real to redshift space transition is algebraic.

The End!

Redshift space

- The observed redshift of a cosmological object has contributions from the Hubble expansion *and* the peculiar velocity.
- We convert z to a distance using a distance-redshift relation.
- Thus in redshift surveys we measure not the true position of objects but their redshift-space position:

$$\mathbf{s} = \mathbf{r} + \hat{r} \cdot \mathbf{v}/(aH)\, \hat{r}$$

- This is both a blessing and a curse:
  - it makes the analysis more complicated, but
  - it gives access to more information.

Moment expansion approach (DF approach) I

Expand $\mathcal{M}$ in powers of $\mathcal{J}$:

$$\Xi_{i_1,\cdots,i_n}(\mathbf{r}) = \langle [1+\delta_a(\mathbf{x}_1)]\,[1+\delta_b(\mathbf{x}_2)]\,\Delta\chi_{ab,i_1}\cdots\Delta\chi_{ab,i_n}\rangle$$

or

$$\widetilde{\Xi}_{i_1,\cdots,i_n}(\mathbf{k}) = (-i)^n\,\frac{\partial\widetilde{\mathcal{M}}_{ab}(\mathcal{J},\mathbf{k})}{\partial\mathcal{J}_{i_1}\cdots\partial\mathcal{J}_{i_n}}\bigg|_{\mathcal{J}=0}$$

then

$$P_{ab}(\mathbf{k}) = \sum_{n=0}^{\infty}\frac{i^n}{n!}k_{i_1}\cdots k_{i_n}\widetilde{\Xi}_{i_1,\cdots,i_n}(\mathbf{k})$$

Perhaps the most straightforward of the methods. Can miss e.g. FoG terms unless one resums afterwards – as in the DF approach. If χ is $\mathcal{O}(\delta)$, keeping only low-order terms gives e.g. Kaiser or linear reconstruction.

Moment expansion approach (DF approach) II

Treating χ as $\mathcal{O}(\delta)$:

$$P_{ab}(\mathbf{k}) = \widetilde{\Xi}_0(\mathbf{k}) + ik_i\,\widetilde{\Xi}_{i,1}(\mathbf{k}) - \frac{1}{2}k_i k_j\,\widetilde{\Xi}_{2,ij}(\mathbf{k}) + \cdots$$

which to lowest order becomes Kaiser (for RSD) or linear reconstruction, e.g.

$$\begin{aligned} P(\mathbf{k}) &= P_{\delta\delta} + 2\mu^2 P_{\delta v} + \mu^4 P_{vv} \\ &= P_L + 2f\mu^2 P_L + f^2\mu^4 P_L \\ &= \left(1 + f\mu^2\right)^2 \; P_L(k) \end{aligned}$$

Smoothing kernel

Expand $\mathcal{M}$ to get $\langle e^{\cdots}\rangle$, $\langle(\delta+\delta')e^{\cdots}\rangle$ and $\langle\delta\delta' e^{\cdots}\rangle$ terms:

$$\left[1 - i\left(\partial_{\lambda_1} + \partial_{\lambda_2}\right) - \partial_{\lambda_1}\partial_{\lambda_2}\right]\left\langle \exp\left[i\lambda_1\delta_1 + i\lambda_2\delta_2 + i\mathcal{J}\cdot\Delta\chi\right]\right\rangle\Big|_{\lambda_1=\lambda_2=0}$$

Use the cumulant theorem on each term ($\cdots$some algebra$\cdots$):

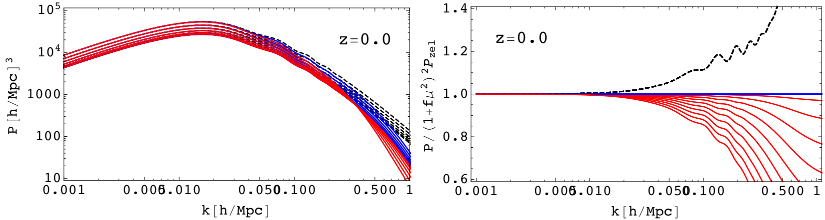
$$\begin{aligned}1+\mathcal{M}(\mathcal{J},\vec{r}) = \exp&\left[\sum_{n=2}^{\infty}\frac{i^n}{n!}\mathcal{J}_{i_1}\cdots\mathcal{J}_{i_n}\left\langle\Delta\chi_{i_1}\cdots\Delta\chi_{i_n}\right\rangle_c\right]\\ &\times\Bigg(1+\left\langle(\delta_1+\delta_2)\,e^{i\mathcal{J}\cdot\Delta\chi}\right\rangle_c\\ &+\left\langle\delta_1 e^{i\mathcal{J}\cdot\Delta\chi}\right\rangle_c\left\langle\delta_2 e^{i\mathcal{J}\cdot\Delta\chi}\right\rangle_c+\left\langle\delta_1\delta_2 e^{i\mathcal{J}\cdot\Delta\chi}\right\rangle_c\Bigg),\end{aligned}$$

(Note translation invariance unbroken!)

Dispersion models and TNS

- The kernel/prefactor, $\exp\left[\sum_{n=2}^{\infty} \cdots\right]$, was historically assumed to be a constant, zero-lag velocity dispersion.
- To get (e)TNS model:
    - Replace exponential with phenomenological, FoG damping term.
    - Expand the exponentials in e.g. $\delta \exp[i\mathcal{J} \cdot \Delta\chi]$ and use 1-loop PT or regPT to compute moments.
    - Add a bias model.

Zeldovich $P(k,\nu)$



Black (dashed) lines are linear theory (with the Kaiser factor), blue lines are Zeldovich times the Kaiser factor and red lines are the full calculation (more damping along the line of sight).