Cosmology in 1 dimension

Martin White

University of California, Berkeley

Lawrence Berkeley National Laboratory

with

Matt McQuinn

University of Washington

"A man grows stale if he works all the time on insoluble problems, and a trip to the beautiful world of one dimension will refresh his imagination better than a dose of LSD."

-- Freeman Dyson

Outline

- The big picture
- Perturbation theory
  - Background.
  - PT in 1D.
  - Successes and limitations.
- Lie'ing with statistics
  - Fun with Lie algebras.

The big picture I

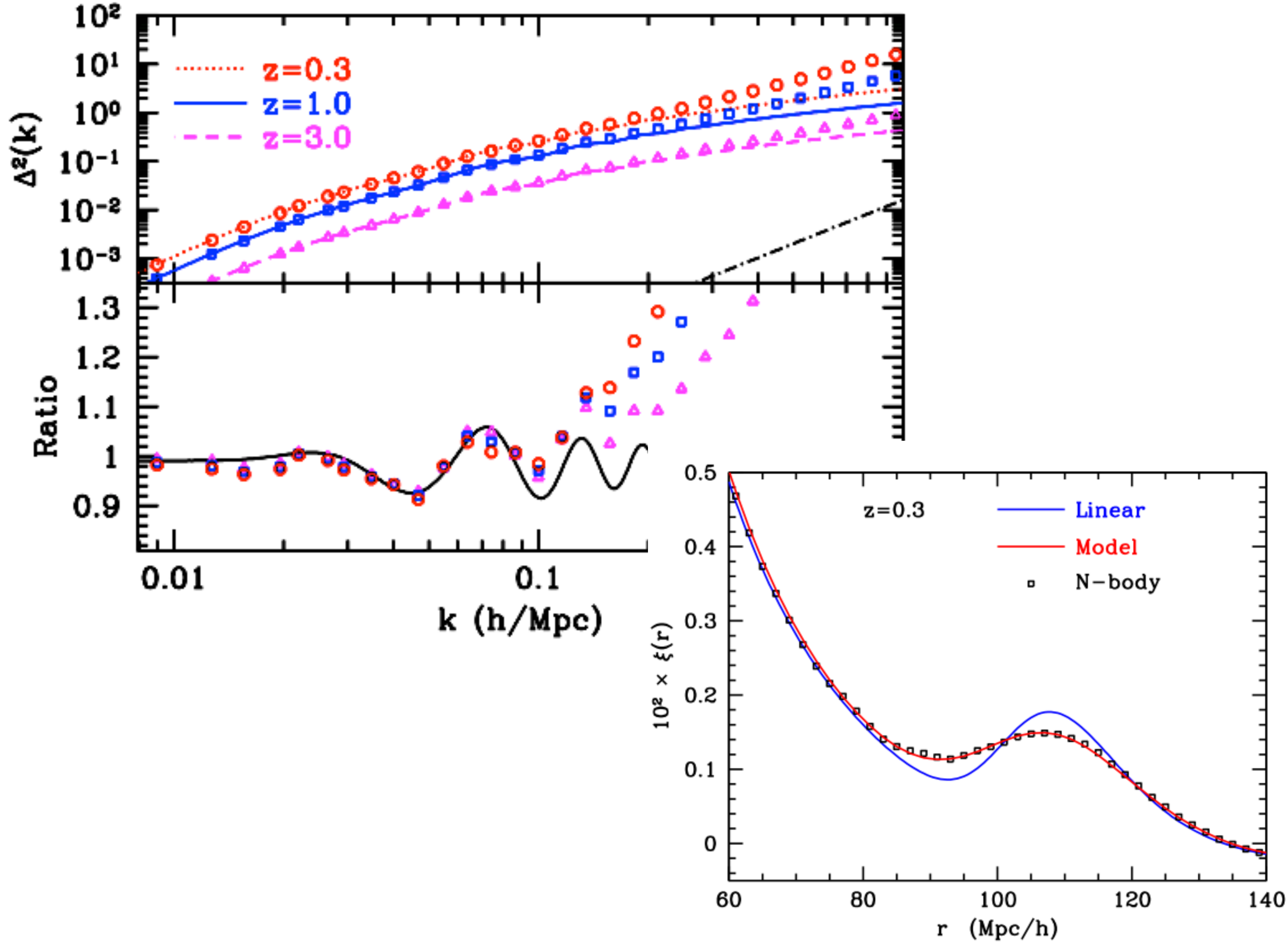
- Something like inflation laid down almost scale-invariant perturbations in the density of all species at early times.
  - Scale invariant means constant metric fluctuations per ln$k$.
  - Through Poisson eqn this is k^4 power in density per ln$k$ (or k^1 in d^3k).
  - Lots of small-scale power – bottom up structure formation.
- These grew through the process of gravitational instability to form all of the structure we see today.

The big picture II

- 14Gyr of evolution shapes the spectrum.
- Growth is a competition between gravity and expansion
  - Depends on laws of gravity.
  - Depends on constituents' properties.
- Linear theory of small perturbations has been impressively validated by studies of CMB anisotropies.
  - Excellent fits to 10^4 d.o.f. with 6 parameters.
- But most of the "modes" are in the quasi- to non-linear regime.

The big picture III

- Since inhomogeneity arose from stochastic fluctuations, all inferences are statistical.
- Compute correlators of the temperature, density, velocity, etc. fields.
- Fluctuations are Gaussian on large scales, so the 2-point functions contain most of the information
  - e.g. $\rho(x)=\langle\rho\rangle[1+\delta(x)]$
  - $\langle\delta(k_1)\delta(k_2)\rangle = (2\pi)^3\,\delta(k_1+k_2)\,P(k)$
- We want models for P(k) that go beyond linear theory ...



z=0.3
z=1.0
z=3.0
Δ²(k)
Ratio
k (h/Mpc)
z=0.3
Linear
Model
N−body
10² × ξ(r)
r (Mpc/h)

Motivation

- There has been a great deal of work recently on (cosmological) perturbation theory.
  - New approaches, new resummation schemes, new renormalization techniques borrowed from QFT.
  - Growing appreciation of the uses and limitations of "standard perturbation theory" (SPT) and resummation schemes.
  - Understanding of RSD, BAO, SSC, beat-coupling, …
- Want to understand these developments (and old ideas) better in a simple context:
- Collection of uniform, parallel, 2D sheets of matter.
  - Problem becomes 1 dimensional (plus time).
  - Significant (!) analytic simplification: can do SPT to ∞ order.
  - Easier to handle numerically with high dynamic range.
  - Many of the features of 3D have close 1D analogues.

Review of cosmological PT

- EOM are both non-linear and non-local.
- PT developed starting in the 60's, reached its present form in the early 90's.
  - Peebles (1980), Juszkiewicz (1981), Goroff++(1986), Makino++(1992), Jain&Bertschinger(1994), Fry (1994).
- We will start with "Eulerian PT".
- Consider **only** dark matter and **assume** we are in the single-stream limit.
  - Describe by density: $\rho = \langle\rho\rangle(1+\delta)$ and velocity, *v.*
  - Velocity field irrotational: specified by divergence, θ.

Equations of motion

Equations of motion, assuming Ω_m=1

$$
\begin{aligned}
\frac{\partial\delta}{\partial\tau} + \vec{\nabla}\cdot\left[(1+\delta)\vec{v}\right] &= 0 \\
\frac{\partial\vec{v}}{\partial\tau} + \mathcal{H}\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} &= -\vec{\nabla}\Phi \\
\nabla^2\Phi &= \tfrac{3}{2}\mathcal{H}^2\delta
\end{aligned}
$$

- Very familiar looking fluid equations
  - means we can borrow methods/ideas from other fields.
- Note the quadratic nature of the non-linearity.
- Proceed by perturbative expansion: $\delta=\delta^{(1)}+\delta^{(2)}+\delta^{(3)}+\ldots$

Go into Fourier space

$$\vec{v}(\vec{q}) \sim \frac{\vec{q}}{q^2}\,\theta(\vec{q})$$

$$\nabla\cdot$$

Product=
Convolution

$$\begin{aligned}
\frac{\partial\delta(\vec{k})}{\partial\tau} + \theta(\vec{k}) &= -\int \frac{d^3q}{(2\pi)^3}\frac{\vec{k}\cdot\vec{q}}{q^2}\,\theta(\vec{q})\delta(\vec{k}-\vec{q}),\\
\frac{\partial\theta(\vec{k})}{\partial\tau} + \mathcal{H}\theta(\vec{k}) + \frac{3}{2}\Omega_m\mathcal{H}^2\delta(\vec{k}) &= -\int \frac{d^3q}{(2\pi)^3}\frac{k^2\,\vec{q}\cdot(\vec{k}-\vec{q})}{2q^2|\vec{k}-\vec{q}|^2}\\
&\times\quad \theta(\vec{q})\theta(\vec{k}-\vec{q}).
\end{aligned}$$

The vector dot products are the equivalent of “vertices” in Feynman diagrams …

Standard perturbation theory

- Develop δ and θ as power series:

$$\delta(\mathbf{k}) = \sum_n a^n \delta^{(n)}(\mathbf{k})$$

$$P(k) = \left\langle \left[\delta^{(1)} + \delta^{(2)} + \cdots\right]^2 \right\rangle$$
$$= P^{(1,1)} + 2P^{(1,3)} + P^{(2,2)} + \cdots$$

- then the $\delta^{(n)}$ can be written

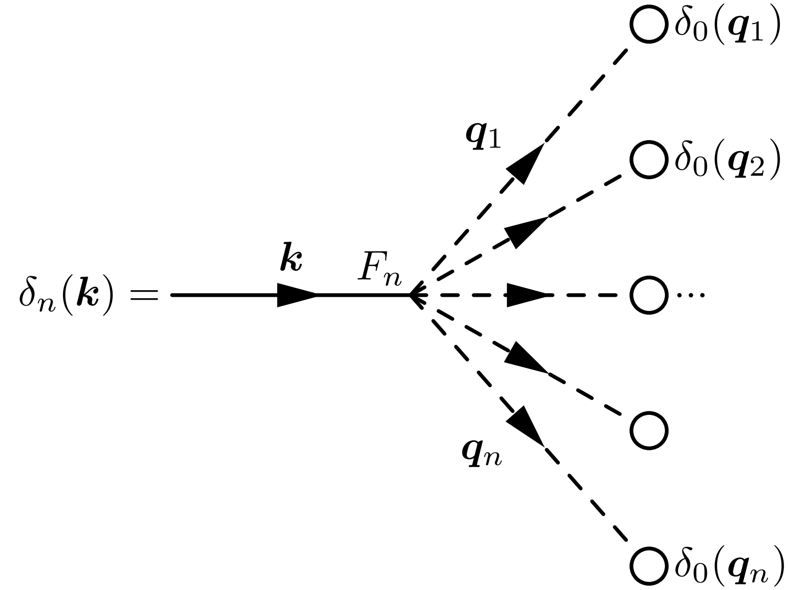
$$\delta^{(n)}(\mathbf{k}) = \int \frac{d^3k_1\, d^3k_2\, \cdots\, d^3k_n}{(2\pi)^{3n}}\, (2\pi)^3 \delta_D\left(\sum \mathbf{k}_i - \mathbf{k}\right)$$
$$\times F_n\left(\{\mathbf{k}_i\}\right)\, \delta_L(\mathbf{k}_1) \cdots \delta_L(\mathbf{k}_n)$$

- with a similar expression for $\theta^{(n)}$.
- The F_n are just ratios of dot products and obey simple recurrence relations.

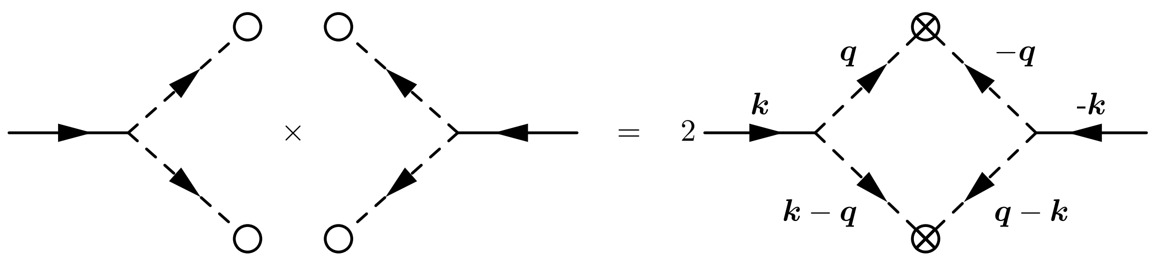
$$F_2(\mathbf{k}_1, \mathbf{k}_2) = \frac{5}{7} + \frac{2}{7}\frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)^2}{k_1^2 k_2^2} + \frac{(\mathbf{k}_1 \cdot \mathbf{k}_2)}{2}\left(k_1^{-2} + k_2^{-2}\right)$$

Perturbation theory: diagrams

Just as there is a diagrammatic short-hand for perturbation theory in quantum field theory, so there is in cosmology:



$$\equiv \quad (2\pi)^3\delta_D(\boldsymbol{q}+\boldsymbol{q}')P_0(q),$$



$$= 2\int \frac{d^3q}{(2\pi)^3} F_2(\boldsymbol{q}, \boldsymbol{k}-\boldsymbol{q})F_2(-\boldsymbol{q}, \boldsymbol{q}-\boldsymbol{k})P_0(q)P_0(|\boldsymbol{k}-\boldsymbol{q}|)$$

Sort-of like QFT

- Can collect δ and θ into an object, ϕ, with an index, write EoM like in QFT.
- Easy to write (Euclidean) path-integral form, generating functions, RG, …

$$\left\langle \phi^a \cdots \phi^b \right\rangle = \int \mathcal{D}\phi_L \ \phi^a[\phi_L] \cdots \phi^b[\phi_L] e^{-\phi_L^i \{P_{ij}^{-1}\} \phi_L^j / 2}$$

$$Z[J] = \int \mathcal{D}\phi_L \ \exp\left(S_0[\phi_L] + J_i \phi^i[\phi_L]\right)$$

- But this is somewhat closer to fluid mechanics and turbulence than QFT, so intuition can be unhelpful at times!
  - As much BBGKY as large-N …

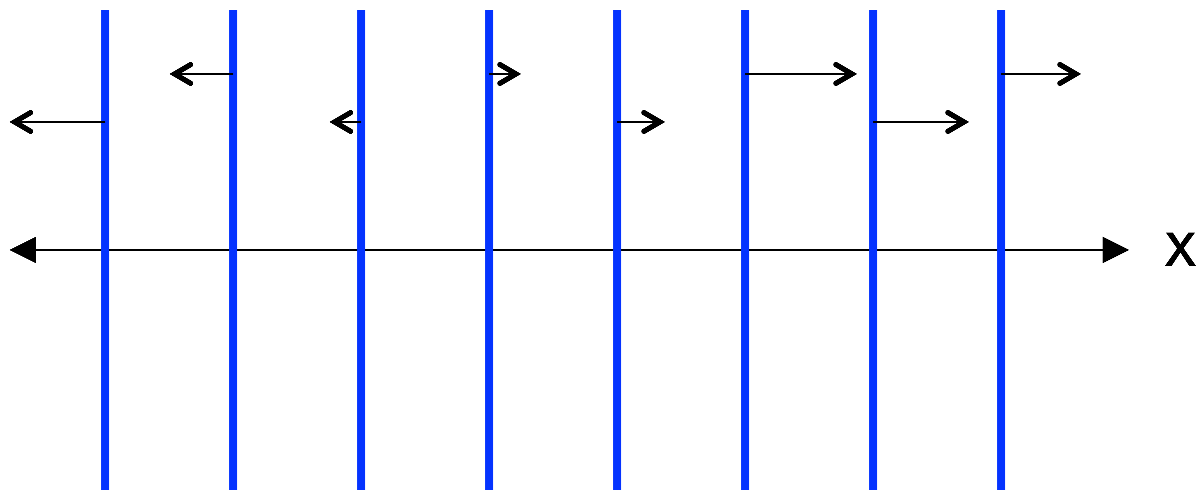
PT in practice

- PT has given us lots of insights and ...
  - It is proving particularly useful for large-scale, high-precision work (e.g. BAO).
  - It may be useful for pushing to smaller scales, and for higher order functions.
- Convergence rather slow.
  - Resummation schemes (but beware symm. breaking).
  - Unfortunately most schemes involves uncontrolled approximations, with no theory of the error.
  - Solutions only valid prior to "sheet" crossing.
  - $P^{1\text{-loop}}(k)$ depends on high-q, non-perturbative modes even at large scales.

Can we gain any intuition on these issues from a toy model?

The setup

(with Ω_m=1 throughout)



$$P_{3D}(k_{\parallel}, \mathbf{k}_{\perp}) = (2\pi)^2 \delta^{(D)}(\mathbf{k}_{\perp}) P_{1D}(k_{\parallel})$$

Since the force on a particle due to a sheet is independent of the distance from the sheet, 1st order Lagrangian PT (Zeldovich) is exact until "sheet crossing" (can also show this analytically).

Equations of Motion

- Eulerian

$$\begin{aligned} \partial_\tau \delta + \theta &= -\partial_x(\delta u) \\ \partial_\tau \theta + \mathcal{H}\theta + 4\pi G a^2 \bar{\rho}\delta &= -\partial_x(u\partial_x u) \end{aligned}$$

$$\delta^{(n)} \sim \int F_n \delta_L(k_1) \cdots \delta_L(k_n)$$

- Lagrangian

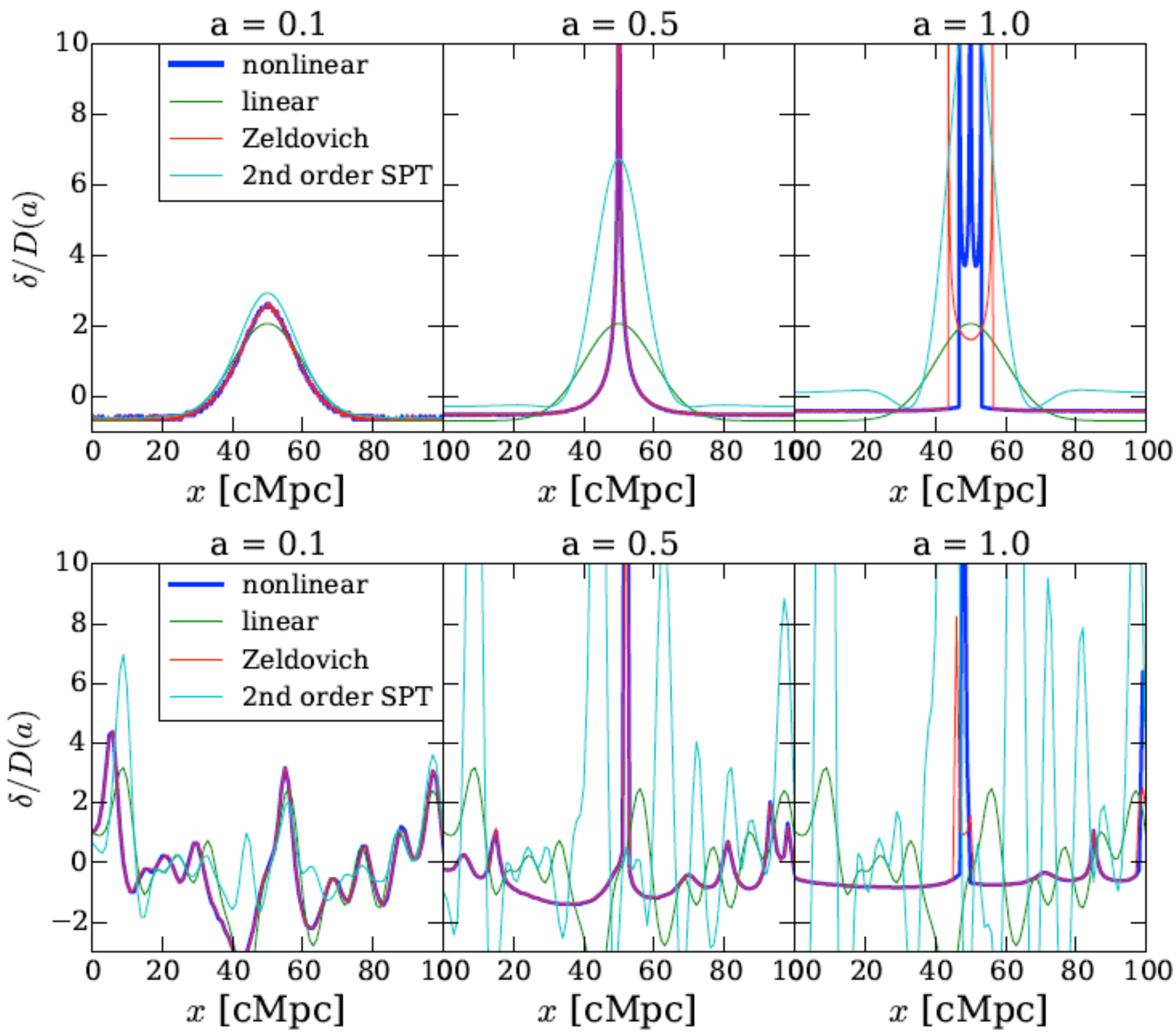
$$\ddot{\Psi}(q) + 2H\dot{\Psi}(q) = -\partial_x \Phi(q + \Psi)$$

$$\Psi^{(n)} \sim \int L_n \delta_L(k_1) \cdots \delta_L(k_n)$$

$x=q+\Psi$

In 1D can show SPT and LPT solutions are identical, to all orders, even though the systems are different!

Evolution



a = 0.1
a = 0.5
a = 1.0
nonlinear
linear
Zeldovich
2nd order SPT
$\delta/D(a)$
x [cMpc]
a = 0.1
a = 0.5
a = 1.0
nonlinear
linear
Zeldovich
2nd order SPT
$\delta/D(a)$
x [cMpc]

Beat coupling/SSC

How does a high k mode respond to a long-wavelength over- or under-density? In the limit that the long wavelength is the size of the survey (or larger) this is known as a "super sample" mode.

$$
\begin{aligned}
\delta^{(2)}(k) &= \int \frac{dk'}{2\pi}\, F_2(k', k-k')\delta_L(k')\delta_L(k-k') \\
&\ni 2F_2(0,k)\,\delta_L(k) \int_{-\epsilon}^{+\epsilon} \frac{dk'}{2\pi}\delta_L(k') \\
&\simeq 2\delta_V\,\delta_L(k) \\
\Rightarrow\ \delta(k) &\simeq [1+2\delta_V]\,\delta_L(k)
\end{aligned}
$$

c.f. $[1+(34/21)\delta_V]$ in 3D.

Power spectra and correlation functions

$$
\begin{aligned}
P_{\rm SPT}^{1-{\rm loop}}(k) &= P_L(k) + \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \left\{ 3 + 4\frac{k-k'}{k'} + \left(\frac{k-k'}{k'}\right)^2 \right\} P_L(k')P_L(k-k') \\
&- k^2 P_L(k) \int_{-\infty}^{\infty} \frac{dk'}{2\pi} \frac{P_L(k')}{k'^2}
\end{aligned}
$$

$$
\xi_{\rm SPT}^{1-{\rm loop}}(x) = \xi_L(x) + \overbrace{3\xi_L^2(x)}^{\text{growth}} + \overbrace{4\xi_L'(x)\int_x^{\infty} dx\, \xi_L(x)}^{\text{dilation}} + \overbrace{\frac{\sigma_{\rm eff}^2}{2}\xi_L''(x)}^{\text{RMS displ.}} + \mathcal{O}(\xi_L^3)
$$

Can look at the response of 2pt fn to long wavelength mode (through the gradient times the large-scale variance), shifts and broadening of the BAO peak, ...

Broadening of the BAO peak

- By far the dominant term (in 1D and in 3D) is the σ^2 term, which broadens the BAO peak.
- Recall near the peak, $\xi \sim 10^{-3}$, $\sigma \sim 10$Mpc, $[\xi'']^{-1/2} \sim 10$Mpc.
- Thus the ξ^2 and $\xi\xi'$ terms are small, but $\sigma^2\xi''$ is $O(1)$.
  - Because Lagrangian theories sum this important term to higher orders, they tend to do better near the BAO peak.
  - The situation in P(k) space is more complicated.

Shifting the BAO peak

- The "dilation" term causes a shift of the BAO peak.
  - $\xi(x[1+\alpha]) \sim \xi(x) + \alpha\, x\xi'(x) + \ldots$
- In overdense regions, the large-scale overdensity acts like a locally closed Universe remapping r to smaller scales.
- Since there is more growth in overdense regions than underdense ones, this leads to a net shift.
- A "separate Universe" approach can predict the coefficient of this term properly in 1D as well as in 3D.

Lagrangian theory (ZA)

$$1+\delta_{\rm LPT}(x)=\int dq\;\delta^D[x-q-\Psi(q)]$$

$$\delta_{\rm LPT}(k)=\int dq\;e^{-ikq}\left(e^{-ik\Psi(q)}-1\right)$$

But Ψ is just a Gaussian random variable … know <e$^{\Psi}$>

$$P_{\rm ZA}(k)=\int dq\;e^{-ik\,q}\left(e^{-k^2\sigma^2(q)/2}-1\right)$$

$$\sigma^2(q)=\langle[\Psi_{\rm ZA}(0)-\Psi_{\rm ZA}(q)]^2\rangle=\int_0^\infty \frac{dk}{\pi}\,\frac{2\,P_L(k)}{k^2}\,(1-\cos[k\,q])$$

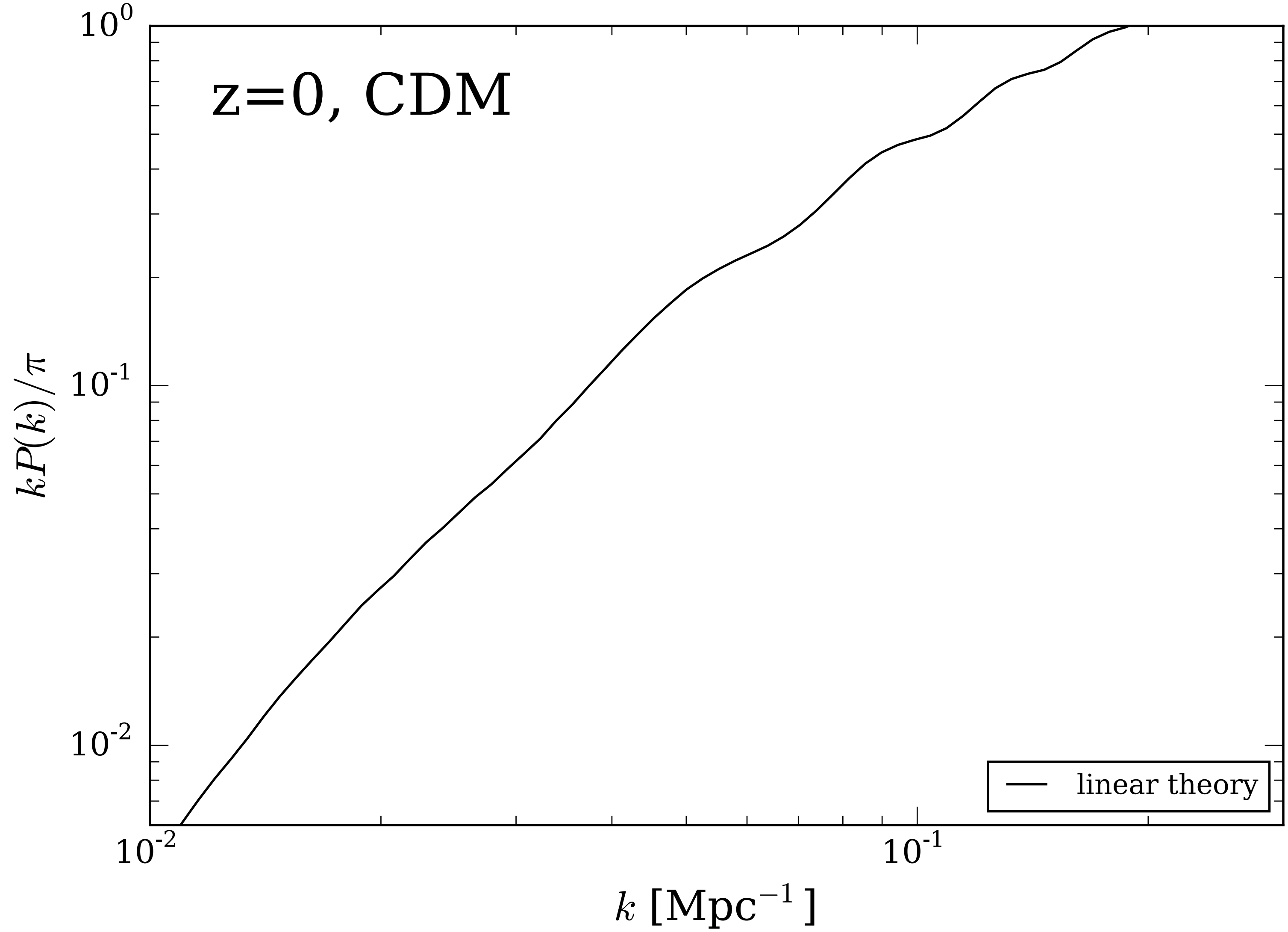
Can generate any order in PT!

$$\begin{aligned}
P_{\rm LPT}(k) &= \int dq\, e^{ik\,q} \left(-\frac{k^2}{2}\sigma^2(q) + \frac{k^4}{8}\sigma^4(q) + \cdots \right), \\
&= P_L + \frac{1}{8}\int dq\, e^{ik\,q}\, \nabla_q^4\, \sigma^4(q) + \mathcal{O}(P_L^3), \\
&= P_L + \frac{1}{8}\int dq\;\, e^{ik\,q} \left[6([\sigma^2]'')^2 + 8([\sigma^2]'[\sigma^2]''') + 2[\sigma^2][\sigma^2]'''' \right] + \mathcal{O}(P_L^3), \\
&= P_L + \int \frac{dk'}{2\pi} \left\{ 3 + 4\frac{k-k'}{k'} + \frac{(k-k')^2}{k'^2} \right\} P_L(k')P_L(k-k') + \cdots\,,
\end{aligned}$$

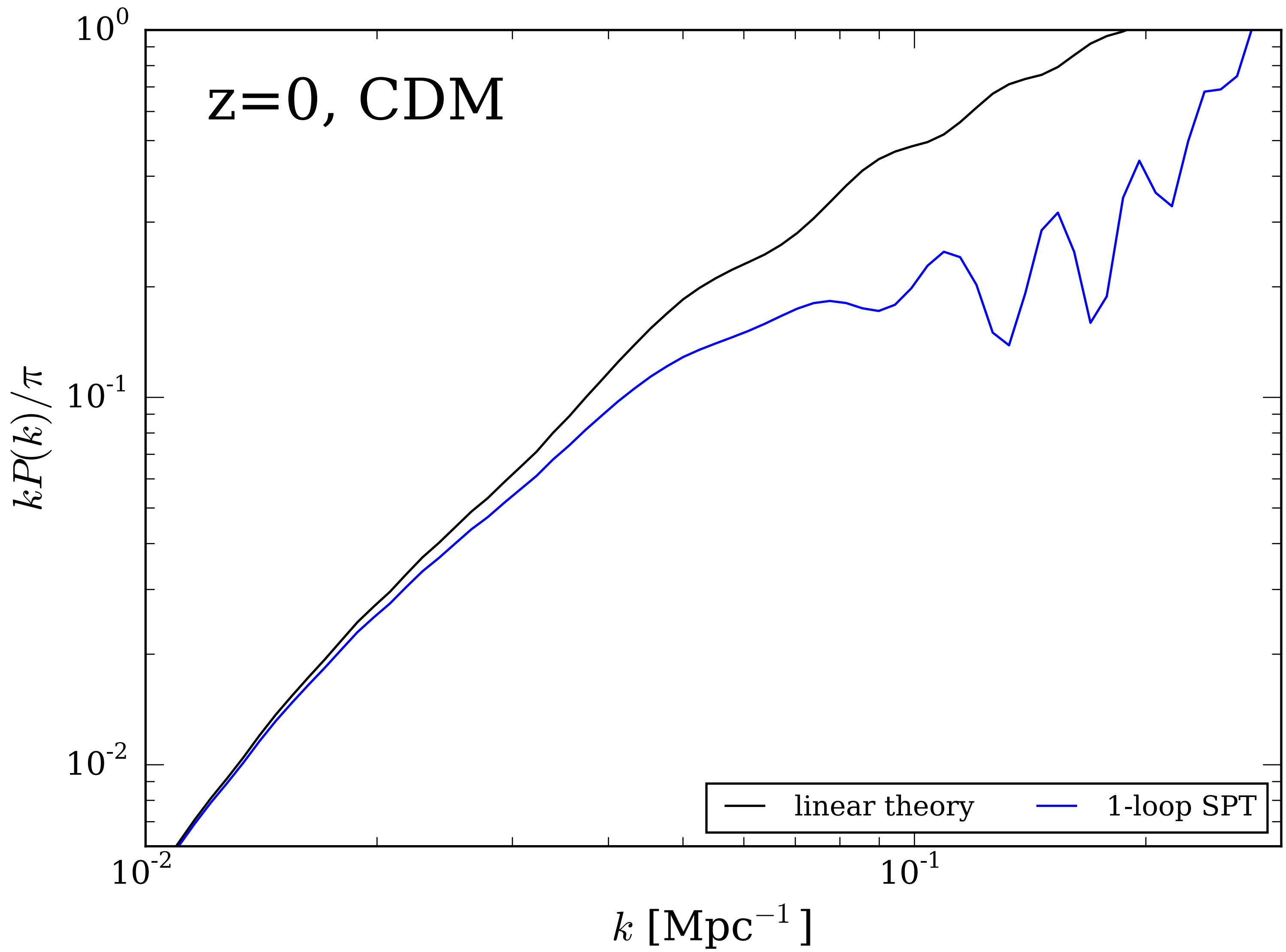
$$\boxed{F_n(k_1,\cdots,k_n) = G_n(k_1,\cdots,k_n) = \frac{1}{n!}\;\frac{k^n}{\prod_{i=1}^n k_i}}$$

Now we can understand common resummation schemes in “standard” perturbation theory, and we can look at the rate of convergence of perturbation theory.

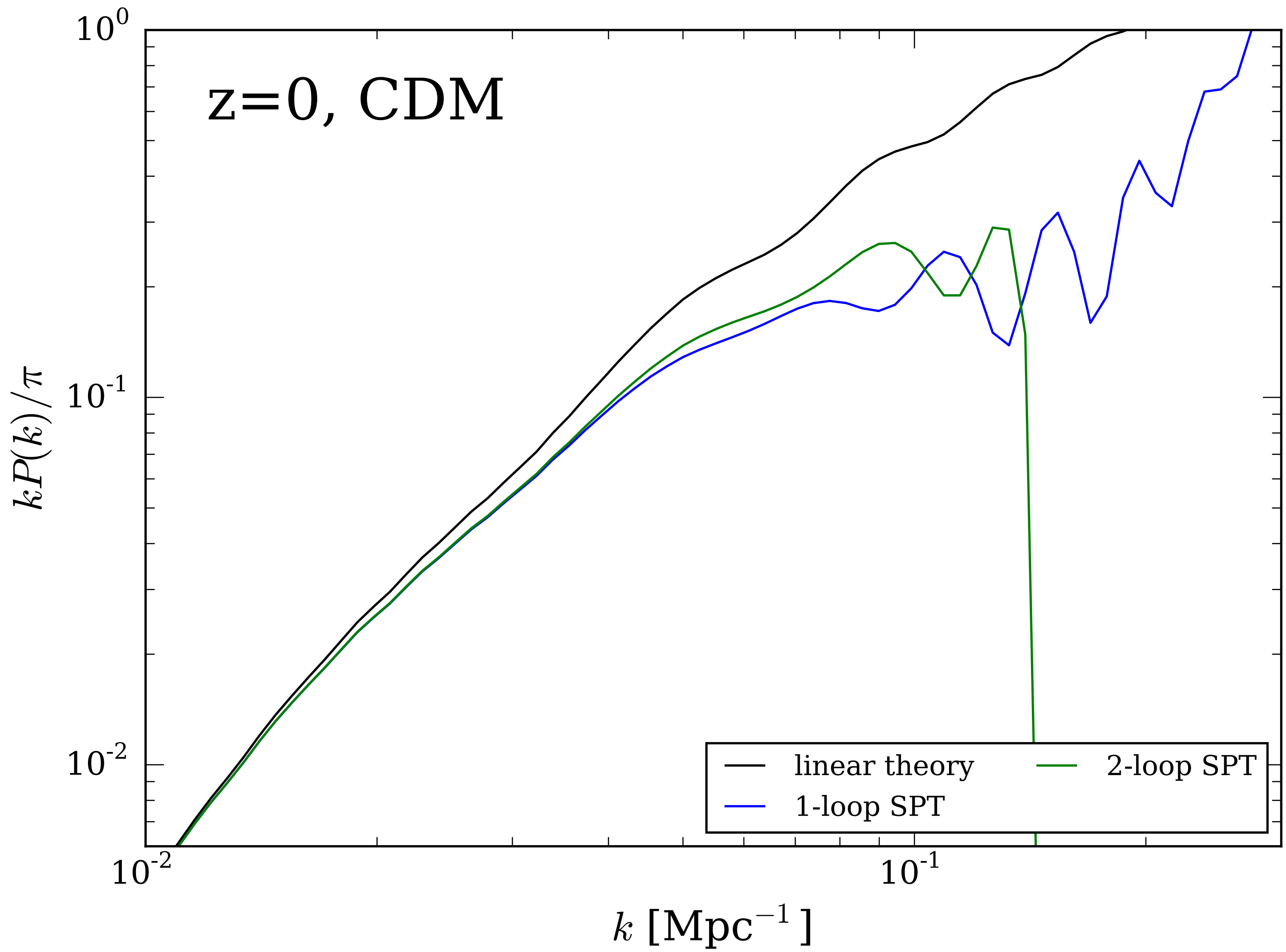
Convergence of PT



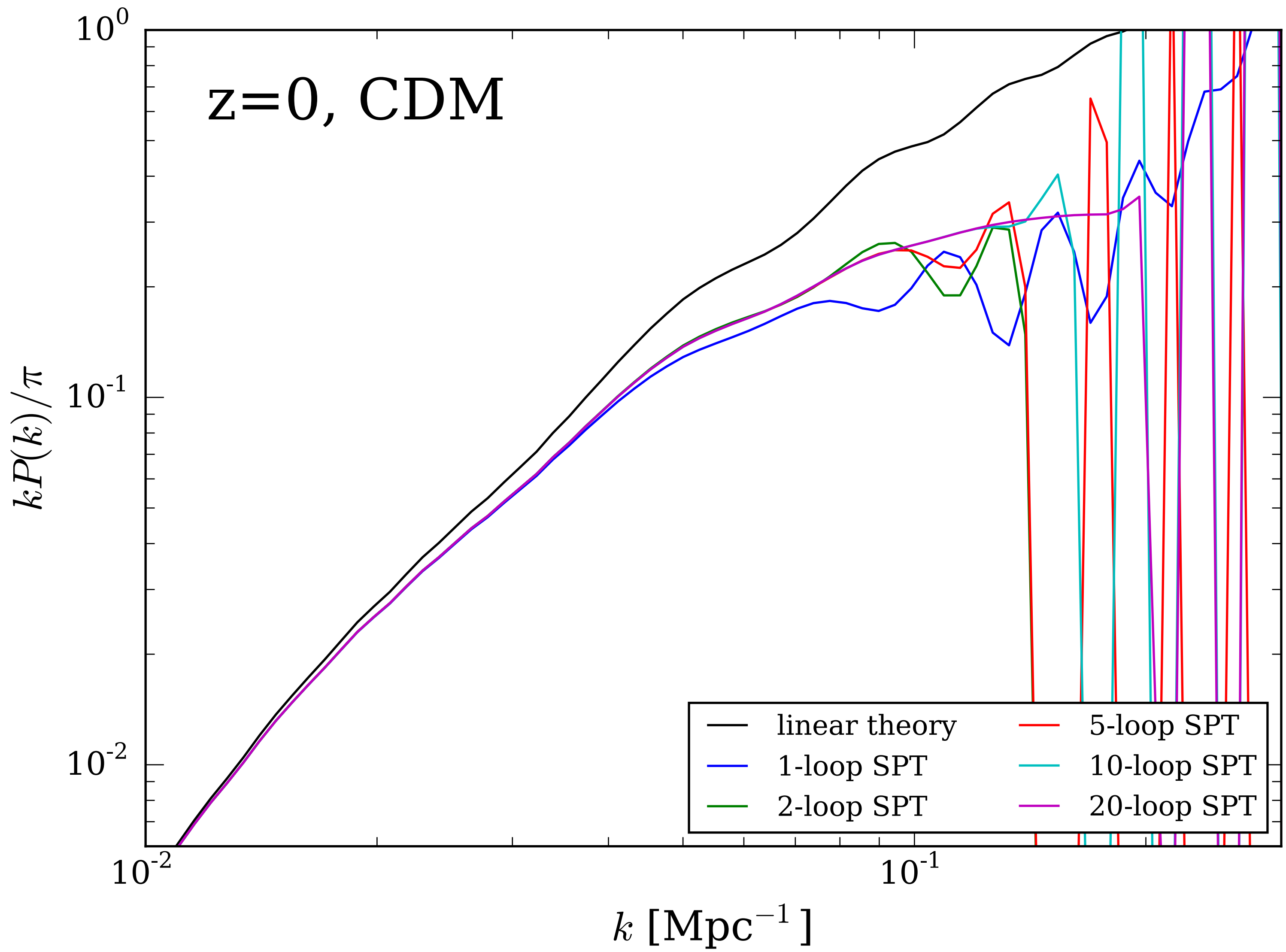
Convergence of PT



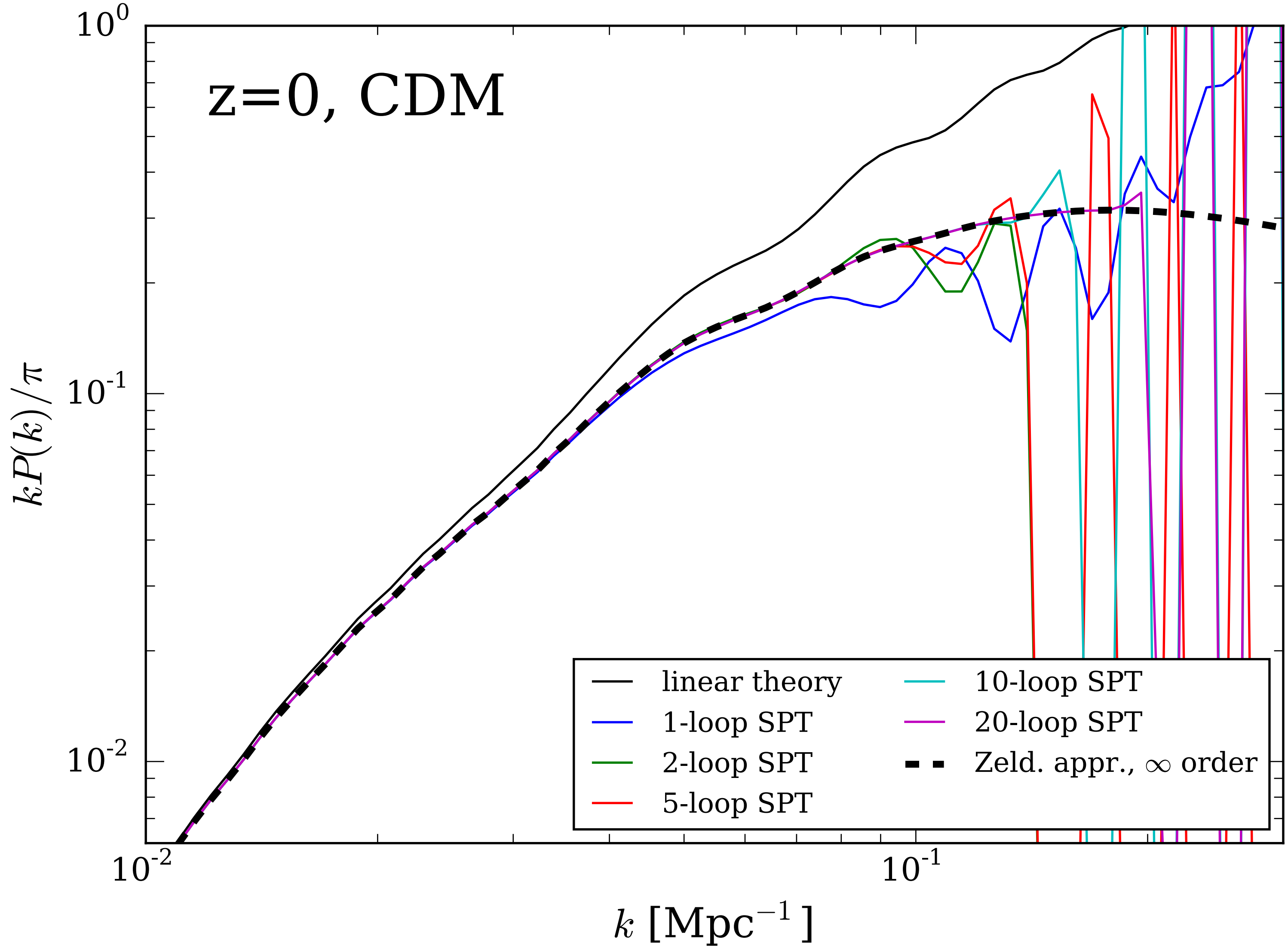
Convergence of PT



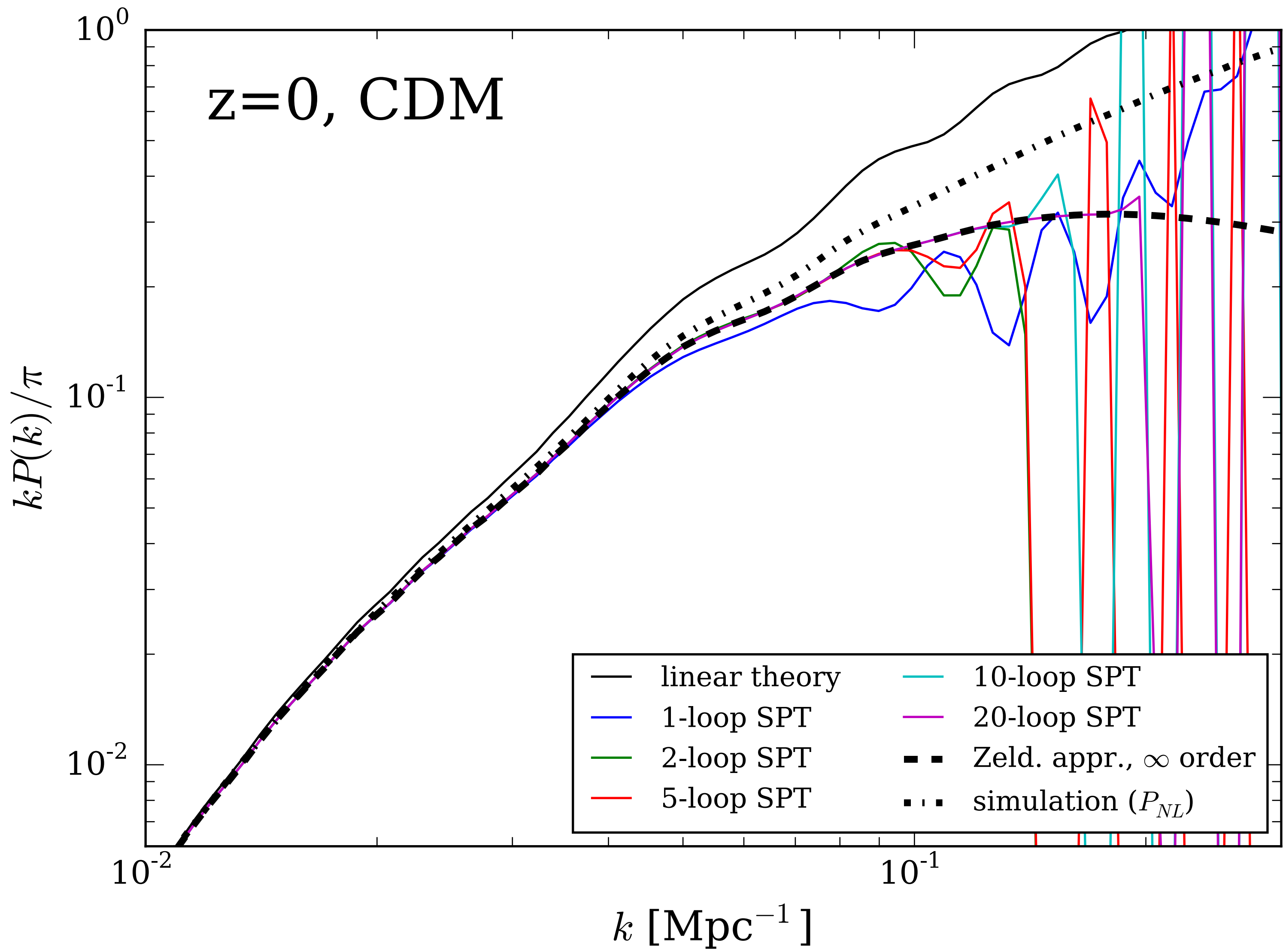
Convergence of PT



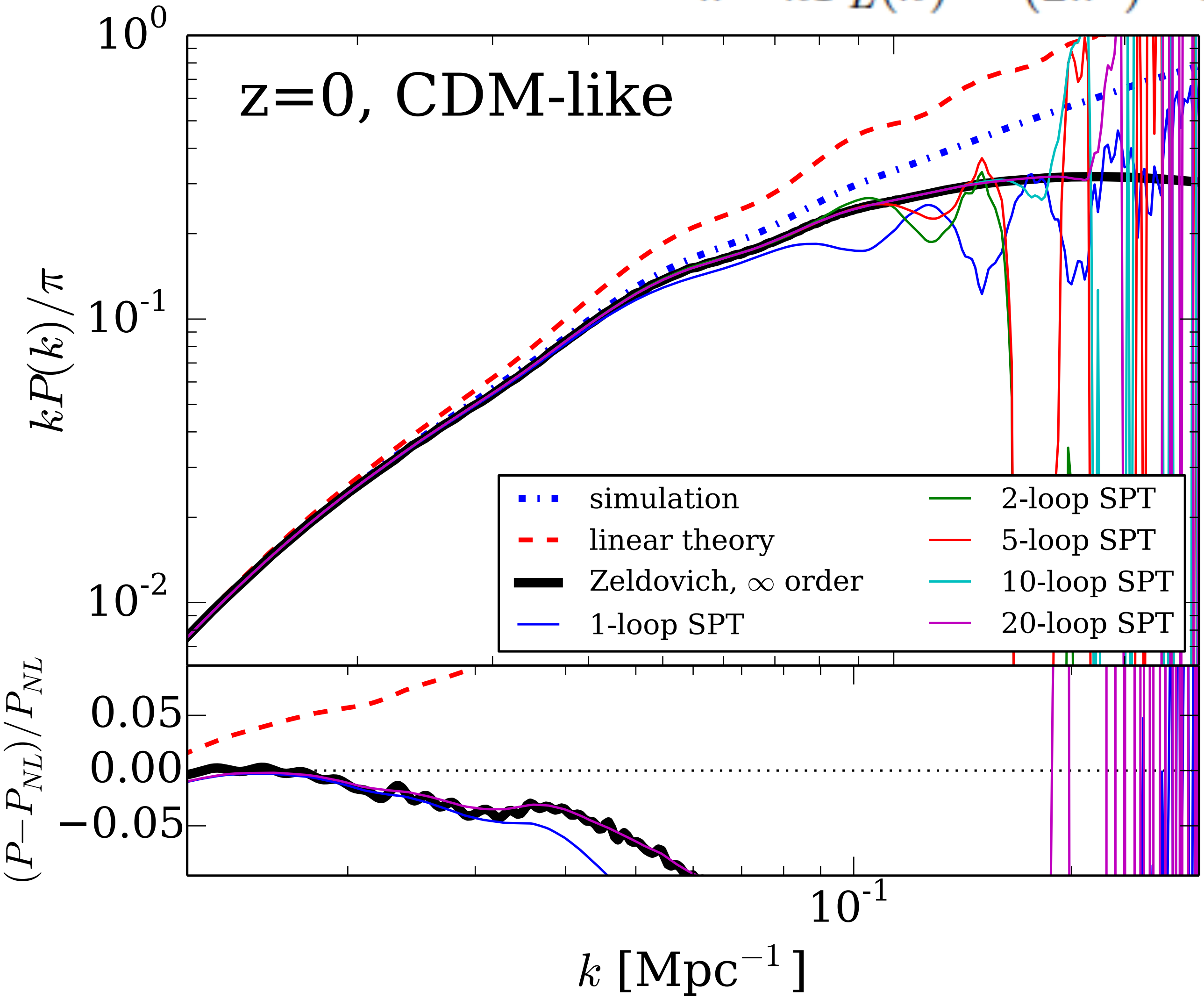
Convergence of PT



Convergence of PT



$$\pi^{-1} k P_L(k) = (2\pi^2)^{-1} k^3 P_{\rm CDM}(k;z)$$



For 1D CDM-like cosmology, standard perturbation theories do not describe evolution on any non-linear scale accurately.

Effective field theory

Traditional perturbation theory treats all scales as if they were perturbative, and the matter field as a perfect fluid. The goal of "EFT" is to overcome these deficiencies.

- "Effective" field theory has a long history in other areas of physics.
  - But cosmology presents some unique features, so beware misleading analogies!
- Basic idea is to write equations only in terms of long-wavelength fields, with no small-scale terms explicitly involved (they've been "integrated out").
- The effects of these small-scale terms then show up as additional terms in effective equations of motion.

$$\mathcal{D}^{(2)}_{\mathrm{lin}}\delta_l = (a\mathcal{H}\partial_a + \mathcal{H})\,\nabla\,(\delta_l u_l) - \nabla\,(u_l \nabla u_l) - \bar{\rho}^{-1}\nabla^2 X_\Lambda$$

What terms are allowed?

- Working to lowest order, X must go as $c_1 J + c_2 k^2 \delta_l$ where J is uncorrelated with δ_l.
  - By mass and momentum conservation, the leading order expansion of J must be k^2.
- At 1-loop we simply integrate against $G(a,a')$, which gives the normal PT terms and just modifies c_i for the "extra" terms.
  - $\delta = \delta^{(1)} + \delta^{(2)} + \delta^{(3)} + c_1' J + c_2' k^2 \delta^{(1)} + \ldots$
- Thus the power spectrum must look like:

$$P^{1-\mathrm{loop}}_{\mathrm{EFTLSS}}(k) = P_{11} + P_{22} + P_{13} + 2\alpha k^2 P_{11} + P_J$$

- Where α can be fit for and P_J goes as k^4 for small k, otherwise unknown (usually dropped for all k).

Lagrangian EFTLSS

- One can also develop a Lagrangian scheme:
  - Hope we can generalize this more easily to include redshift space distortions, bias, reconstruction, etc.

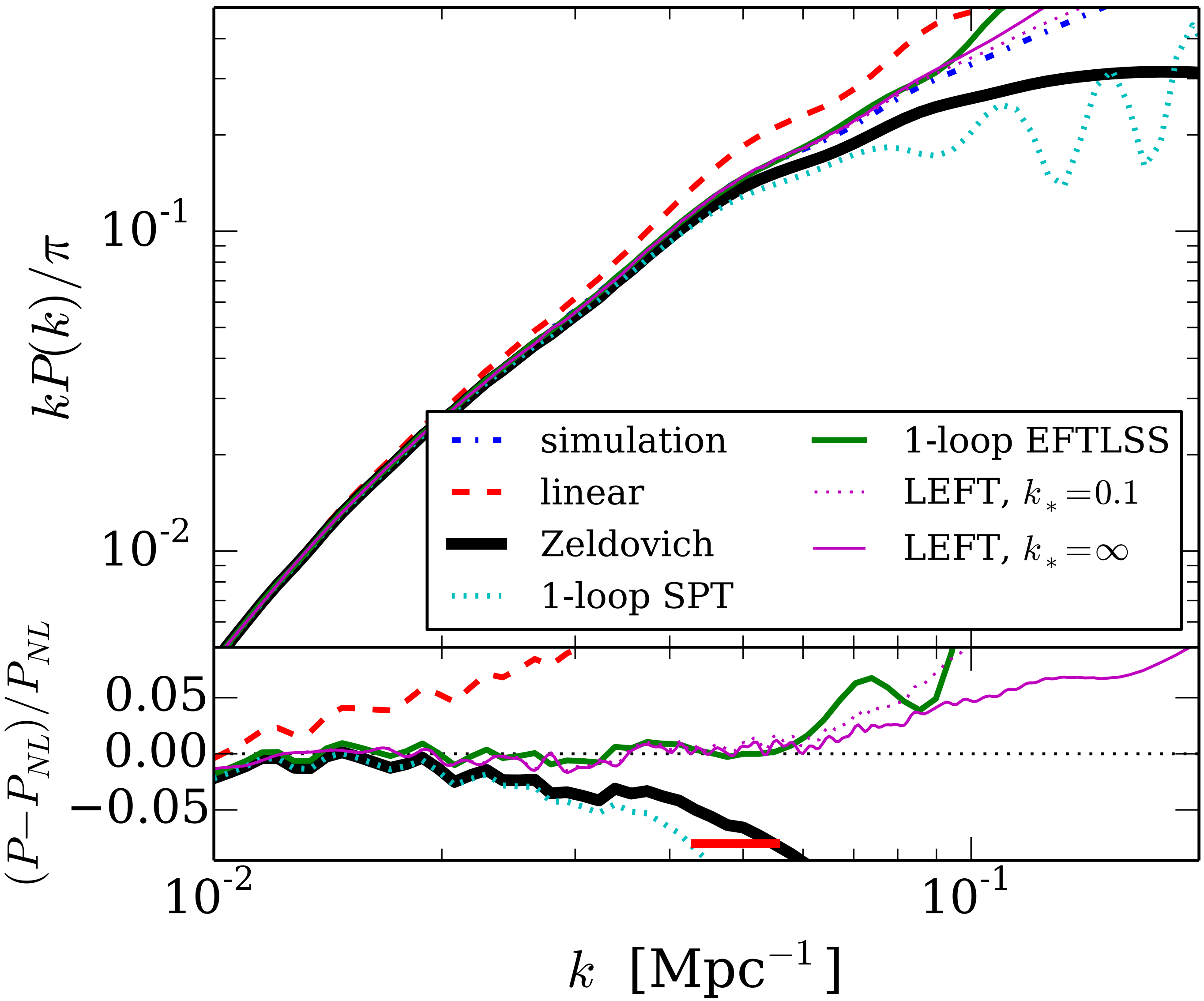
$$\Psi = \Psi_l + \Psi_s \approx \Psi_l + 2\alpha\nabla\delta_l + \nabla J$$

$$P(k) = \int dq\ e^{-ikq}\left[e^{-(1/2)k^2\sigma_{\rm eff}^2} - 1\right]$$

$$\sigma_{\rm eff}^2(q) = \sigma^2(q) + 4\alpha\left[\xi(0) - \xi(q)\right] - 2\nabla^2\xi_J(q)$$

Have choice of keeping terms exponentiated or consistently expanding by order … numerically not much difference. Expression for the correlation function is easy …

Comparison



10^{-1}
10^{-2}
$kP(k)/\pi$
simulation
linear
Zeldovich
1-loop SPT
1-loop EFTLSS
LEFT, $k_*=0.1$
LEFT, $k_*=\infty$
$(P-P_{NL})/P_{NL}$
0.05
0.00
−0.05
10^{-2}
10^{-1}
k [Mpc^{-1}]

Going to 3D

- Many of the same lessons carry across to 3D.
  - Structure of the theory is mathematically identical, mostly it's just coefficients in front of terms which change (modestly).
  - Zeldovich is no longer exact, but it's still pretty good!
  - Effects of shell crossing are somewhat smaller (c.f. caustic formation in spherical collapse vs. "the real world").
- It is possible to use these insights to develop a 3D Lagrangian EFT.
  - e.g. Porto++(2014); Vlah++(2015a,b)

Conclusions (so far)

- Cosmological PT in 1D has some nice features.
  - Easy to simulate, easy to calculate.
  - Can do SPT to ∞ order.
  - Algebra for common methods easier to understand.
  - Close analogs to many 3D effects/situations.
- Can prove SPT converges … to the wrong answer.
- Can understand Fourier vs. Configuration and Euler vs. Lagrange more easily.
- EFTLSS is much simpler in 1D.
  - Easier to see analytically what's happening.
  - Dramatic improvement for power-law models (where symmetry is really helping).
- Nice "toy" problem for understanding PT.

Lie'ing with statistics

- When comparing theory and observations you need to compute a likelihood function.
- If you're lucky, the central limit theorem tells you the likelihood is Gaussian.
- So you need the theory, μ, data, *d*, and a covariance matrix, *C*.

$$\mathcal{L}(\vec{p}) \propto |C(\vec{p})|^{-1/2} \exp\left[-\frac{1}{2}\left(d_i(\vec{p}) - \mu_i(\vec{p})\right)^T C_{ij}^{-1}\left(d_j(\vec{p}) - \mu_j(\vec{p})\right)\right]$$

Interpolation

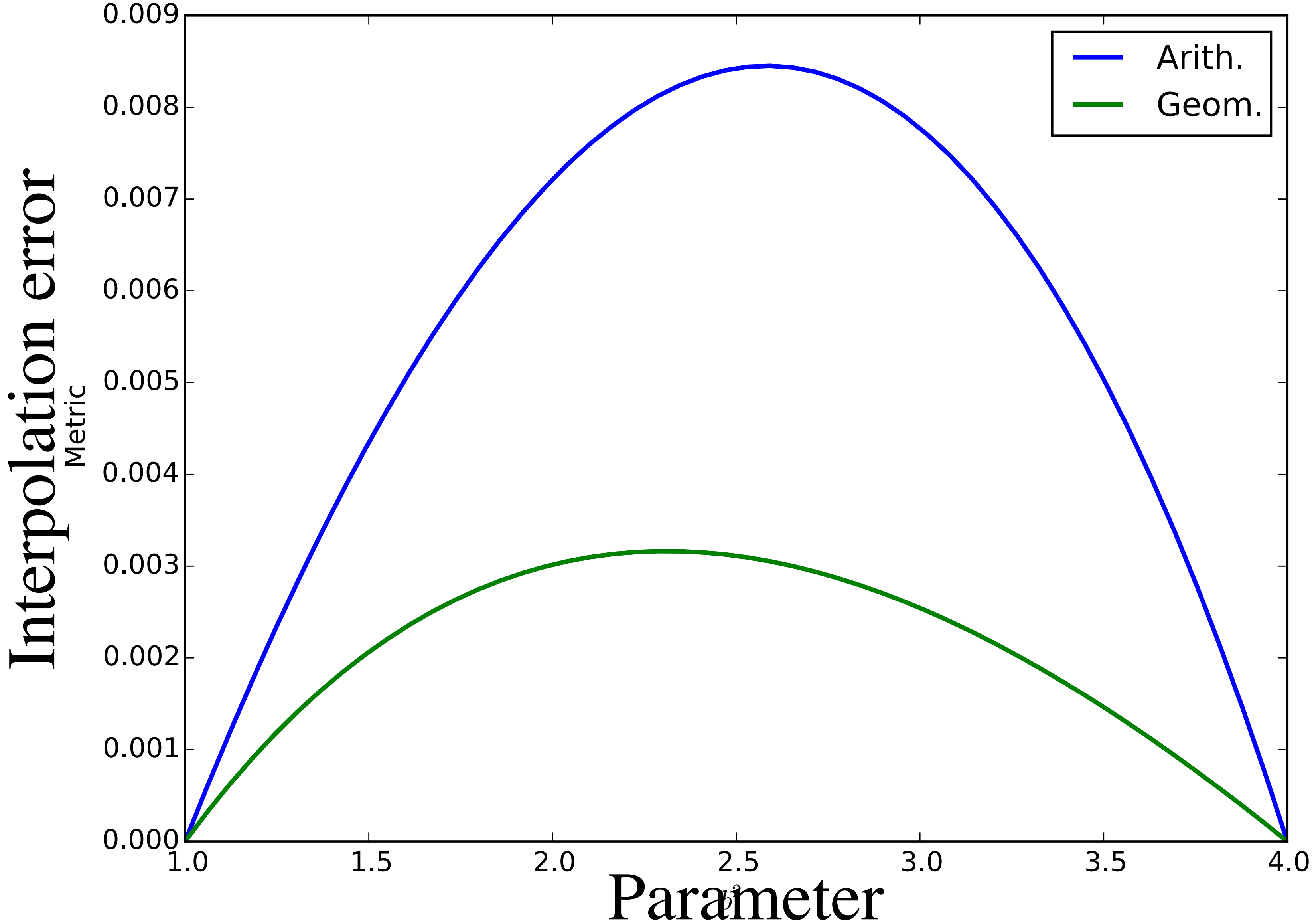
- Often computing *C* is hard/expensive.
- If compute it at a set of points, {p}, can I interpolate to other values?

- *C* is a symmetric, positive-definite matrix (SPD).
- These form a subset (actually a convex cone), P, of GL(n), which is a Lie group and thus a manifold.
  - The tangent space at the identity is the Lie algebra, *g*.
  - GL(n) acts transitively on P.
  - There is a natural inner product (Frobenius).
  - Have geodesics: exp(tg) for t in [0,1].

Parallel transport

- Recall the average, x, of a set $\{x_i\}$ minimizes distance: $\Sigma_i \|x-x_i\|^2$
- Since our tangent vectors (group generators) are related to group elements by exp, it's no surprise that lengths are "logarithmic".
- To interpolate from C_0 at t=0 to C_1 at t=1 we can do
  - $C(t) = C_0[C_0^{-1}C_1]^t = C_0^{1/2}[C_0^{-1/2}C_1C_0^{-1/2}]^t C_0^{1/2}$

Example in 1D: 40x40 matrices



Computer graphics

- In fact this is precisely the scheme used in computer graphics to interpolate camera movement!!
  - Using quaternions in place of rotation matrices.
- Also used in MR imaging and medicine.
- The generalization to multiple dimensional interpolation is slightly subtle, since matrices don't commute, but doable.
  - There is some fun math and math history about this problem for matrices.
- Allows all sort of "distance based" algorithms to be applied to SPD matrices …

The End